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**Research Article**

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# The generalized canonical equations $K_1, K_7, K_{16}, K_{27}$. The REFORM method, the invariance principal method, the matrix expansion method and $G$-transform. The main stochastic canonical equations $K_{100}, ..., K_{106}$ and the estimators $G_{55}, ..., G_{58}$ of the MAGIC (Mathematical Analysis of General Invisible Components)

**Abstract:** The generalized version of the stochastic canonical equations $K_1, K_7, K_{16}$ and $K_{27}$ are founded under the generalized Lindeberg condition with the help of which the main MAGIC estimators $G_{55}, G_{56}, G_{57}G_{58}$ are found

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Enrico Fermi said, I remember my friend Johnny von Neumann used to say, with four parameters I can fit an elephant, and with five I can make him wiggle his trunk. Freeman Dyson, A meeting with Enrico Fermi, NATURE, **427**(2004) 297

## 1 Introduction

Among all the applications of random matrices, their application in statistics occupies the main place. Their role was especially evident in multivariate statistical analysis, where the probability density of eigenvalues of Gaussian random matrices was first found. In 1986, a new theory was developed, the so-called $\mathbf{MAG^3I^6C}$-theory(Mathematical Analysis of { **General, Generalized, Global**}, {*Inadequate, Indefinite, Invisible, Incorrect, Inappropriate, Incomprehensible etc.*}, Components), in which we do not require the existence of probability densities of observed vectors $\vec{\xi}_k$ and consider the case when the dimension $m$ of the vectors $\xi_k$ are comparable to the number of observations $n$. In this theory, we required that the components of the vectors $R^{-1/2}\vec{\xi}_k$, where $R$ is the covariance matrix, be independent and for the first time we have found consistent and asymptotically normal estimators of some important expressions(See MAGIC estimators $G_i, i = 1, ..., 54$ in [2,3]) In this article we do not require this condition and find a consistent MAGIC estimators $G_{55}, G_{56}, G_{57}G_{58}$ for the normalized traces of the resolvent of a covariance matrix $R$ and of a matrix $A$. These estimators are the main one in our analysis and with its help we can find consistent estimators of other functions of the entries of the covariance matrix $R$ and $A$.

Many people warned us to be careful with large dimension in statistics. Some scientists said even more cruelly about large dimension. Many people very often quote the Richard's Bellman words "curse of dimensionality." In MAGIC we have overcomed some difficulties and have proved for the first time the consistency of new estimators $G_{55}, G_{56}, G_{57}G_{58}$.

In this article, we generalize the stochastic canonical equations to more complex matrices

$$A_n+\sum_{k=1,\dots,m}\{B_n^{(k)}+C_n^{(k)}\Xi_n^{(k)}D_n^{(k)}\}\{B_n^{(k)}+C_n^{(k)}\Xi_n^{(k)}D_n^{(k)}\}^*$$

with the help of which we find the consistent estimators of the matrices based on the independent observations of random matrices. Without loss of generality, we consider a special case of such matrices when k is equal to one. Particular examples of such matrices have been considered in a great number of literatures, but only in the papers [1–4] the canonical equations for matrices whose entries have different variances and for which the generalized Lindeberg condition is satisfied were found for the first time. We use the basic equation $K_{16}$ [3]of the MAGIC theory under the assumption that the vector columns of random matrices $\Xi_{n\times n}$ are stochastically independent and do not impose any conditions on the stochastic dependence of the components of its column vectors: let $\vec{\eta}_k, k=1,...,n$ be independent, identically distributed random vectors of dimension $m_n$ and $A_{m_n}$ is a Hermitian matrix. Then the basic equation of the MAGIC theory is

$$Q_{m_n}(z)=\left\{-I_{m_n}z+A_{m_n}+\mathbf{E}\frac{\vec{\eta}_1\vec{\eta}_1^{\,*}}{1+n^{-1}\vec{\eta}_1^{\,*}Q_{m_n}(z)\vec{\eta}_1}\right\}^{-1},\Im z>0.$$

The proof was obtained in [1-4] and is based on the following equality

$$P_{m_n}(z)-G_{m_n}(z)=P_{m_n}(z)\left(\frac{1}{n}\sum_{k=1}^{n}\vec{\eta}_k\vec{\eta}_k^{\,*}-\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left\{\frac{\vec{\eta}_k\vec{\eta}_k^{\,*}}{1+n^{-1}\vec{\eta}_k^{\,*}\{\mathbf{E}\,G_{m_n}(z)\}\vec{\eta}_k}\right\}\right)Q_{m_n}(z),$$

where

$$G_{m_n}(z)=[-I_{m_n}z+A_{m_n}+\frac{1}{n}\sum_{k=1}^{n}\vec{\eta}_k\vec{\eta}_k^{\,*}]^{-1},$$

and

$$P_{m_n}(z)=\left[-I_{m_n}z+A_{m_n}+\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left\{\frac{\vec{\eta}_k\vec{\eta}_k^{\,*}}{1+n^{-1}\vec{\eta}_k^{\,*}\{\mathbf{E}\,G_{m_n}(z)\}\vec{\eta}_k}\right\}\right]^{-1}.$$

Using this equation $K_{16}$, we find an estimator $G_{55}$ for the trace of the resolvent of the covariance matrix $R_{m_n}$ based on the independent observations $\vec{x}_k, k=1,...,n$ of a random vector $\vec{\eta}$.

So, the basic estimator $G_{55}$ of the MAGIC-theory is

$$G_{55}(\alpha+\mathrm{i}x)=\left\{I_{m_n}(\alpha+\mathrm{i}x)+\frac{1}{n}\sum_{k=1}^{n}(\vec{x}_k-\hat{\vec{x}}_{(k)})(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\tilde{\theta}_k^{-1}\right\}^{-1},$$

where $x$ is a real variable and $\alpha>0$ is a certain constant, the complex random variables $\tilde{\theta}_k, \Re\tilde{\theta}_k>0, k=1,...,n$ satisfy the following system of stochastic canonical equations $K_{100}$:

$$\tilde{\theta}_k+\frac{1}{n}(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\left\{I_{m_n}(\alpha+\mathrm{i}x)+\frac{1}{n}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\hat{\vec{x}}_{(k)})(\vec{x}_j-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\tilde{\theta}_j^{-1}\right\}^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})=1,k=1,..,n,$$

where

$$\hat{\vec{x}}_{(k)}=n^{-1}\sum_{j=1,j\neq k}^{n}\vec{x}_j.$$

This estimator $G_{55}(\alpha+\mathrm{i}x)$ differs from the standard estimator $m_n^{-1}\mathrm{Tr}\,[I_{m_n}(\alpha+\mathrm{i}x)+\hat{R}_{m_n}]^{-1}$ of the trace of the resolvent $m_n^{-1}\mathrm{Tr}\,[I_{m_n}(\alpha+\mathrm{i}x)+R_{m_n}]^{-1}$ which is still used for centuries in numerous literature and in numerous applied problems. Under some conditions $G_{55}(\alpha+\mathrm{i}x)$ is a consistent estimator, namely

$$p \lim_{m_n, n\to\infty, m_n n^{-1}\to\gamma} \left\{ m_n^{-1} \mathrm{Tr}\, G_{55}(\alpha + \mathrm{i}x) - m_n^{-1} \mathrm{Tr}\left[I_{m_n}(\alpha + \mathrm{i}x) + R_{m_n}\right]^{-1} \right\} = 0.$$

**Remark 1.1.** *A few remarks about the notation. In different formulas different constants are denoted by one letter c, the norm of a matrix* $||A_n||$ *is its maximum singular eigenvalue, the limit in the mean of a random variables* $\lim_{n\to\infty} \mathbf{E}\,|\xi_n| = 0$ *is denoted as* $\text{l.i.m.}_{n\to\infty}\xi_n = 0$*, the constants tending to zero when* $n \to \infty$ *are denoted as* $\epsilon_n$, $I_n$ *is the identity matrix. The notation* $A_n > 0_n$ *for the Hermite matrix* $A_n$ *means that it is nonnegatively defined. Some parameters we will omit but when we need them we will write them again in matrix notations. In different non-overlapping sections we will sometimes use the same letters, but this coincidence will not affect the meaning of our proofs.*

# 2 We follow forty-years old axiomatics of the Mathematical Analysis of General Invisible Components (MAGIC)

Thanks to MAGIC axiomatics, we avoid the eternal search for the definition of the probability of an event.[2,4] In MAGIC axiomatics, we first define a quality criterion for the estimation of the probability based on any measure $\mu$ that can be replaced by an empirical measure $\hat{\mu}_n$.

# 3 How can we avoid the main contradiction in probability theory?

The most important thing in MAGIC axiomatics is that we can avoid using an unknown, mythical, non-existent, incomprehensible probability measure.[2,4] Instead of such a measure, we use an empirical measure, and most importantly, we will use this measure to evaluate the proximity of the model and system. In the abstract theory of probability it is required the existence of the unique probability space $(\Omega, \mathcal{F}, \mathbf{P}\,)$ and in the corresponding statistical theory of von Mises it is required the existence of a limit of empirical probability measures $\hat{\mathbf{P}}_n$, so that in some sense $\lim_{n\to\infty} \hat{\mathbf{P}}_n = \mathbf{P}$ . These words "some sense" is very delicate and usually means a vicious sophistic circle of estimation: we use the empirical probability $\hat{\mathbf{P}}_n$ using the unknown probability $P$ as its criterion of accuracy and we have to start estimating $P$ again and so on. In MAGIC we replace such condition with the condition where instead of one abstract probability space we have a sequence of certain abstract probability spaces. But, of course, we can dream a little and assume that we have observations of some probability measure under certain conditions, which are still covered by a dense veil of secrecy in many studies. In this case, with a certain choice of quality criterion (again under certain unknown conditions!) a miracle occurs and the empirical measure $\hat{\mu}_n$ in such a quality criterion approaches this probability measure $\mu$ as the number of observations increases.

# 4 Axiom 1. A sequence of running models $M_n$ of a system $S$ is given

We start from the fact that there is a sequence of systems $S_l, l = 1.2, ...$ that we consider as an objective reality, for example we have the system **probability of an event** or the sequence of the systems of linear algebraic equations (SLAE)$A_l\vec{x}_l = \vec{b}_l, l = 1.2, ...$, but we also assume that the system can be unobservable. For example, there is no a system of flipping a coin in a completely unpredictable way.

We assume that the dimension $m_n$ of the model $M_{m_n}$ of a system $S_l$ can increase together with the number $n$ of observations of a system $S_l$. Analysing many practical problems we can confirm that indeed $n$ depends on $m_n$ and cannot grow arbitrarily fast as $m_n$ itself increases. It is supposed there is a sample of observations $x_1, x_2, \dots, x_n$ of a system $S_l$. For theoretical analysis of models we consider the sequence of observations $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}, \quad n = 1, 2, \dots$ of a systems $S_l$ (random arrays). We assume that the dimension $m$ of theoretical vector-observations can change, when the number of observations $n$ itself increases, i.e. we assume that we have a sequence of models $M_{m_n}, n = 1, 2, \dots$.

## 5 Axiom 2. The dimension of an estimated functional $\varphi(S)$ of a system $S$ is fixed

In MAGIC we have some difficulties when we estimate the system $S$, because we apply this analysis when we have a number of observations which is almost the same as the number of unknown parameters.[2,4] From the analysis of many statistical problems we can conclude that instead of estimating the system $S$, we must estimate some functional $\varphi(S)$. For example, we do not need to estimate the matrix $A_n$ in SLAE but we need to find $\vec{c}A_n^{-1}\vec{b}_n$. Therefore, in this analysis, we assume that the dimension (the number of unknown parameters) of the estimated characteristics $\varphi(S)$ of the system $S$ will not change, when the number $m_n$ of parameters of the models $M_n$ of the system $S$ increases. This assumption is met in many practical problems.

## 6 Axiom 3. The $G$-condition (the uncertainty principle) is given and the existence of the "critical point" is assumed

The numbers of unknown parameters $m_n$ of running models and the number of observations $n$ of a system $S$ satisfy the $G$-condition:

$$\limsup_{n\to\infty} f(m_n, n) \leq \hbar < \infty,$$

where $f(m_n, n)$ is some positive function increasing in $m_n$ and decreasing in $n$. In most cases $f(x, y)$ can be chosen to be $f(m_n, n) = m_n n^{-1}$ or more often $f(m_n, n) = m_n n^{-2}$.(see[2,4]) The constant $\hbar$ depends on the system $S$ and is called the "critical point". This means that if

$$\limsup_{n\to\infty} f(m_n, n) > \hbar$$

then it is impossible to find a consistent estimator of a certain functional $\varphi(S)$ of the system $S$. A similar constant known as Planck's constant has already been encountered in quantum mechanics. That is, these concepts can be explained in some cases using the one particular MAGIC $G$-condition when $f(m_n, n) = n^{-2}m_n$: $\lim_{n\to\infty} m_n n^{-2} = \gamma, 0 < \gamma < \infty$, where $m_n$ is the number of unknown parameters of a model, for example the number of the entries of a covariance matrix. By the way, this condition has a deep philosophical meaning: as we increase observations $n$ of a system, we are sometimes forced to change the dimension $m$ of our models. This condition has already been encountered in probability theory in the Bernoulli scheme under the conditions of Poisson's Theorem when the probability $p$ of success depends on the number $n$ of trials and satisfies the condition $\lim_{n\to\infty} p_n n = \gamma, 0 < \gamma < \infty$. It would seem that this is nonsense. In fact, this is not so, but simply using this condition it is very convenient to consider the problems when fulfilling this condition for a fixed $n$. For those people who do not understand the axiomatics of MAGIC, remember the Poisson Theorem for the Bernoulli testing scheme.

# 7 Axiom 4. The sequence of probability spaces is given. The principle of running probability spaces $(\Omega_n, \mathcal{F}_n, \mathrm{P}_n)$

In MAGIC we introduce a sequence of certain abstract probability spaces $(\Omega, \mathcal{F}_n, \mathbf{P}_n), n = 1, 2, ...$. The corresponding empirical distribution functions $\hat{\mathbf{P}}_{m_n}$ do not converge in general with distribution functions $\mathbf{P}_n$ , although some functional (such as expectation $\vec{a} = \int \vec{x} d\hat{\mathbf{P}}_{m_n}(\vec{x})$ or the covariance matrices $\int (\vec{x} - \vec{a})(\vec{x} - \vec{a})^T d\hat{\mathbf{P}}_{m_n}(x)$ of random vectors) converges to the vector and covariance matrix for the corresponding measure $\mathbf{P}_n$ of the sequence of probability spaces.

Instead of the convergence of the empirical distribution functions $\hat{\mathbf{P}}_{n,m_n}$, we use the principle of running distributions functions $\mathbf{P}_n$ to which the empirical distribution functions $\hat{\mathbf{P}}_{m_n}$ converge[2,4]:

$$p \lim_{n,m_n: n\to\infty, m_n\to\infty} |\hat{\mathbf{P}}_{n,m} - \mathbf{P}_n| = 0.$$

A good example that confirms this principle is the limit theorems of random matrix theory, where, as a rule, empirical spectral distribution functions $\mu_{m_n}(x)$ do not converge to the limit function but converge to a sequence of non-random distribution functions $F_{m_n}(x)$: for any $\epsilon > 0$

$$\lim_{n,m_n: n\to\infty, m_n\to\infty} \mathbf{P}_{m_n}\{\sup_x |\mu_{m_n}(x) - F_{m_n}(x)| < \epsilon\} = 1.$$

It is obvious that in this case we have wider application of our theory.

# 8 Axiom 5. A certain quality characteristic exists

The most important aim in MAGIC theory is to define a quality characteristic of the sequence of models $M_{m_n}(\omega)$. We consider the following quality characteristic for $M_m(\omega)$-models in MAGIC, in which we can choose a measure with the goal of simplifying calculations [2,4]

$$I(S, \hbar) = \sup_s \lim_{n,m_n\to\infty, m_n n^{-2}\to\hbar} \int_\Omega \|\varphi(S) - \varphi(M_{m_n}(\omega)\| \, d\mathbf{P}_s(\omega),$$

where $\|\cdot\|$ is a distance between the system $S$ and the model $M_n(\omega)$, $\mathbf{P}_s$ is a sequence of probability measures, $\phi(S)$ is a functional of a system $S$..

As the reader sees a measure in MAGIC quality criteria there can be any and there is no indication of how this measure should be chosen if there is not an empirical measure available. If there are no assumptions that we are dealing with observations of random variables, we use the quality criterion in the form

$$I(S, \hbar) = \sup_\omega \lim_{n,m_n\to\infty, m_n n^{-2}\to\hbar} \|\varphi(S) - \varphi(M_{m_n}(\omega)\|.$$

Let's see how things are with the choice of the quality criterion in other sciences, for example, in the theory of estimating parameters. Here the universally accepted method is the least-squares method chosen to simplify calculations. But it does not follow that this criterion is the best.

# 9 Axiom 6. Feedback control also exists

If the criterion quality characteristic $I(S, \hbar)$ exceeds a certain constant, which we call the "confidential constant" then we have to reach one of two conclusions: 1). Our probability measure is wrong. Then we can try to change $\mathbf{P}_m$ by $\hat{\mathbf{P}}_n(m)$, an empirical measure. 2). Our model $M_m$ is wrong.Then we have to find a new, more precise, model $M_{m+1}$ and calculate new quality characteristic $I(S,\hbar)$ choosing measure $\mathbf{P}_m$ and model $M_{m+1}$. Therefore, we have to include the feedback control $C(S-M_m)$ in our analysis.

Representing the axioms symbolically we say that MAGIC is specified if the following eight objects are given

$$\Big\{\hbar,\, S,\, \varphi(S),\, \Omega,\, \mathcal{F},\, \mu_n,\, I(S,\hbar), C(S-M_m)\Big\}.$$

In the following sections we present a results which we can use for the deriving the main estimators of MAGIC. For some of them it can be proven that under certain conditions they are consistent and sometimes even asymptotically Normal(see[2,4]). Note that these estimators can significantly decrease the number of observations required to solve many practical problems.

# 10 Determinants of $2\times 2$ block matrices

Let all quadratic matrices $A,B,C,D$ have the same size and let $A$ and $D$ be non-singular. Obviously

$$\begin{Bmatrix} I & 0 \\ -CA^{-1} & I \end{Bmatrix}\begin{Bmatrix} A & B \\ C & D \end{Bmatrix} = \begin{Bmatrix} A & B \\ 0 & D-CA^{-1}B \end{Bmatrix}.$$

Then

$$\det\begin{Bmatrix} A & B \\ C & D \end{Bmatrix} = \det A \det\left\{D-CA^{-1}B\right\}, \det\begin{Bmatrix} A & B \\ C & D \end{Bmatrix} = \det D \det\left\{A-BD^{-1}C\right\}. \tag{10.1}$$

# 11 The usefulness of the perturbation formulas for block matrices

The Frobenius formula for non degenerated matrices:

$$\begin{Bmatrix} A_{m_1\times m_1} & B_{m_1\times m_2} \\ C_{m_2\times m_1} & D_{m_2\times m_2} \end{Bmatrix}^{-1} = \begin{Bmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}BH^{-1} \\ -H^{-1}CA^{-1} & H^{-1} \end{Bmatrix}, \tag{11.1}$$

where $H=D-CA^{-1}B$, the matrices $A$ and $H$ are non-degenerate, and to simplify this formula, we omit the matrix indices of their dimensions.

Consider matrices $A_n+\{B_n+C_n\Xi_nD_n\}\{B_n+C_n\Xi_nD_n\}^*$ (see [5]) and new its transform, i.e. their normalized trace of the resolvent with the positive parameter $\alpha>c>0$ and any parameter $x$

$$f(\alpha+\mathrm{i}x) := \frac{1}{n}\mathrm{Tr}\left[I_n(\alpha+\mathrm{i}x)+A_n+\{B_n+C_n\Xi_nD_n\}\{B_n+C_n\Xi_nD_n\}^*\right]^{-1} \tag{11.2}$$

and the matrix $A_n$ is the non negative defined Hermitian matrix.

We also will consider the main $G$-transform of MAGIC

$$\mathrm{Tr}\left[I_n(\alpha+\mathrm{i}x)+G_n(\alpha+\mathrm{i}x)\right]^{-1}, \tag{11.3}$$

where $G_n(\alpha+\mathrm{i}x)$ is not an analytical random matrix.

**Remark 11.1.** *We call these transforms (11.2) and (11.3) with a positive parameter $\alpha>c>0$ as the $G$-transform. It differs from the Stiltjes transform with a complex parameter $z, \Im z>0$ and sometimes its limit expressions cannot be analytically continued to the complex plane. That is why the formula for its inverse transform for its limit expressions is much more complicated than the inverse formula for the Stiltjes transform. Therefore, to emphasize that this transform is much more complicated, we give* *it a new name G-transform. But of course, by virtue of analyticity of the resolvent on the parameter* $\alpha, \alpha > 0$ *we can in some cases analytically continue the trace of the resolvent on the complex plane* $z = t + \mathrm{i}\epsilon, \epsilon > 0$.

**Remark 11.2.** *Without loss of generality we assume that the matrices* $C_n, D_n$ *are non-singular. Otherwise, we can instead of these matrices* $C_n, D_n$ *to consider regularized nondegenerate matrices* $C_n(\epsilon), D_n(\epsilon)$, *where* $\epsilon$ *a small parameter chosen in such a way that the minimum singular eigenvalues of the matrices* $C_n, D_n$ *are greater than a positive constant. For example, we can choose* $C_n(\epsilon) = U_n(\Lambda_n + I_n\epsilon)V_n, \epsilon > 0$, *where* $U_n, V_n$ *are Unitary matrices and* $\Lambda_n$ *is the diagonal matrix of singular eigenvalues. Let us denote the obtained G-transforms* $\mathbf{E}\, f(\alpha + \mathrm{i}x, C_n, D_n, ..), \mathbf{E}\, f(\alpha + \mathrm{i}x, C_n(\epsilon), D_n(\epsilon), ..)$. *Then under the conditions imposed on the entries of random matrices in our theorems we will obtain*

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\sup_{\alpha\geq c>0,x}|\mathbf{E}\, f(\alpha+\mathrm{i}x, C_n, D_n, ..) - \mathbf{E}\, f(\alpha+\mathrm{i}x, C_n(\epsilon), D_n(\epsilon), ..)| = 0.$$

Using notations $L_n(t) = C_n^{-1}(I_nt + A_n)(C_n^*)^{-1}$, $P_n = C_n^{-1}B_nD_n^{-1}$ and $M_n = D_nD_n^*$ and equality (11.1) we get for $t > 0$

$$\begin{aligned}
f(t) \quad &:= \frac{1}{n}\operatorname{Tr} C_n^{-1}(C_n^*)^{-1}[C_n^{-1}(I_nt + A_n)(C_n^*)^{-1} + \{C_n^{-1}B_nD_n^{-1} + \Xi_n\}D_nD_n^*\{C_n^{-1}B_nD_n^{-1} + \Xi_n\}^*]^{-1}\\
&= \frac{\partial}{\partial t}\frac{1}{n}\ln\det[C_n^{-1}(I_nt + A_n)(C_n^*)^{-1} + \{C_n^{-1}B_nD_n^{-1} + \Xi_n\}D_nD_n^*\{C_n^{-1}B_nD_n^{-1} + \Xi_n\}^*]^{-1}\\
&= \frac{\partial}{\partial t}\frac{1}{n}\ln\left\{\det[C_n^{-1}(I_nt + A_n)(C_n^*)^{-1} + \{P_n + \Xi_n\}M_n\{P_n + \Xi_n\}^*]\right\}\\
&= \frac{\partial}{\partial t}\frac{1}{n}\ln\left\{\det\left[\begin{Bmatrix}\mathrm{i}M_n^{-1} & P_n^*\\ P_n & \mathrm{i}C_n^{-1}(I_nt + A_n)(C_n^*)^{-1}\end{Bmatrix} + \begin{Bmatrix}0 & \Xi_n^*\\ \Xi_n & 0\end{Bmatrix}\right]\right\}\\
&= \frac{1}{n}\operatorname{Tr}\begin{Bmatrix}0 & 0\\ 0 & \mathrm{i}C_n^{-1}(C_n^*)^{-1}\end{Bmatrix}\left[\begin{Bmatrix}\mathrm{i}M_n^{-1} & P_n^*\\ P_n & \mathrm{i}L_n(t)\end{Bmatrix} + \begin{Bmatrix}0 & \Xi_n^*\\ \Xi_n & 0\end{Bmatrix}\right]^{-1}\\
&= \frac{1}{n}\operatorname{Tr} Q_{2n}\operatorname{Tr} R_{2n},
\end{aligned}\qquad(11.4)$$

where $R_{2n} = [\Gamma_{2n} + H_{2n}]^{-1}$,

$$Q_{2n} = \begin{Bmatrix}0 & 0\\ 0 & \mathrm{i}C_n^{-1}(C_n^*)^{-1}\end{Bmatrix}, \Gamma_{2n} = \begin{Bmatrix}\mathrm{i}M_n^{-1} & P_n^*\\ P_n & \mathrm{i}L_n(t)\end{Bmatrix}, H_{2n} = (h_{ij}) = \begin{Bmatrix}0 & \Xi_n^*\\ \Xi_n & 0\end{Bmatrix}.$$

We see that our problem has been reduced to the problem of finding spectral functions of the sum of a non-random matrix $\Gamma_{2n}$ and a Hermite random matrix $H_{2n}$, which will be very convenient for finding estimators of matrices $\Gamma_{2n}$ using empirical mean $(2n)^{-1}\sum_{j=1,...,2n}X_{2n}^{(j)}$ of observations of the matrix $H_{2n}$.

**Remark 11.3.** *We consider this transform under the condition that* $t > 0$ *when the matrix* $A_n$ *is the non negative defined Hermitian matrix. But if* $A_n$ *is the Hermitian matrix and its eigenvalues can be negative, we assume that* $t > max_{k=1,...,n}|\lambda_k\{A_n\}| + c, c > 0$. *However, this condition does not limit the generality of our proofs, since in the final formulas for resolvents of matrices, due to their analyticity in* $t > max_{k=1,...,n}|\lambda_k\{A_n\}| + c, c > 0$, *we can continue them analytically for all values* $z = t + \mathrm{i}\epsilon, \epsilon \neq 0$.

# 12 The proof of non degeneracy of a matrix $\Gamma_{2n} + H_{2n}$ and new additional parameter $\alpha > 0$

If the eigenvalues of the matrices $M_n^{-1}, (C_nC_n^*)^{-1}, A_n$ are bounded from below and above by some positive constants and $t > 0$ we have that the eigenvalues of the matrices $M_n^{-1}, L_n$ are grater than a some constant. Therefore the singular values of the matrix $\{\Gamma_{2n}\}^{-1}$ are bounded by some constant. Then it is obvious that the matrix $\Gamma_{2n} + H_{2n}$is nondegenerate. This gives us an opportunity to introduce in (11.4) a new additional parameter $\alpha > 0$ :

$$f(t,\alpha) = \frac{1}{n} \operatorname{Tr} Q_{2n} R_{2n}(t,\alpha), \tag{12.1}$$

where

$$Q_{2n} = \begin{Bmatrix} 0 & 0 \\ 0 & \mathrm{i} C_n^{-1}(C_n^*)^{-1} \end{Bmatrix}, R_{2n}(t,\alpha) = \Big[\Gamma_{2n}(\alpha) + H_{2n}\Big]^{-1}, t > 0,$$

$$\Gamma_{2n}(\alpha) = \begin{Bmatrix} \mathrm{i}\alpha M_n^{-1} & P_n^* \\ P_n & \mathrm{i}\alpha L_n(t) \end{Bmatrix}, H_{2n} = (h_{ij}) = \begin{Bmatrix} 0 & \Xi_n^* \\ \Xi_n & 0 \end{Bmatrix}.$$

## 13 The main statement. Canonical equations $K_1$ and $K_7$

We generalize the proof of the canonical equations $K_1$ and $K_7$(see [4]) under the generalized Lindeberg condition which based on the REFORM method and the invariance principle for the resolvents of random matrices.

**Theorem 13.1.** ([4, Chapters 1 and 7]) *Assume that the random entries $\xi_{ij}^{(n)}$, $i = 1,...,n$, $j = 1,...,n$, of the matrix $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i=1,...,n}^{j=1,...,n}$ are independent for any $n$,*

$$\mathbf{E}\, \xi_{ij}^{(n)} = 0, \quad \mathbf{Var}\, \xi_{ij}^{(n)} = \sigma_{ij}^{(n)}, \lim_{n\to\infty} \max_{i,j=1,...,n} \sigma_{ij}^{(n)} = 0,$$

$$\sup_n \max_{\substack{p=1,\ \dots\ ,n,\\ l=1,\ \dots\ ,n}} \left[ \sum_{j=1}^n \sigma_{pj}^{(n)} + \sum_{i=1}^n \sigma_{il}^{(n)} \right] < \infty, \tag{13.1}$$

*the generalized Lindeberg's condition is satisfied, i.e., for every $\tau > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1,...,n,j=1,...,n} \mathbf{E} \left[\xi_{ij}^{(n)}\right]^2 \chi\left\{\left|\xi_{ij}^{(n)}\right| > \tau\right\} = 0, \tag{13.2}$$

*the singular eigenvalues of the matrices $A_n, C_n, D_n$ are bounded from below and above by some positive constants, $A_n$ is the Hermitian matrix, matrix $B_n$ has bounded singular values*

$$\mu_n \{x, A_n + (B_n + C_n \Xi_n D_n)(B_n + C_n \Xi_n D_n)^*\} = n^{-1} \sum_{k=1}^n \chi\{\omega :\ \lambda_k < x\},$$

*and $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of the random matrix $A_n + (B_n + C_n \Xi_n D_n)(B_n + C_n \Xi_n D_n)^*$.*

*Then, with probability one for almost all $x$,*

$$\lim_{n\to\infty} |\mu_n (x, A_n + \{B_n + C_n \Xi_n D_n\}\{B_n + C_n \Xi_n D_n\}^*) - F_n (x)| = 0, \tag{13.3}$$

*where $F_n(x)$ is the non random distribution function in $x$ whose G-transform satisfies relation for all $t > 0$*

$$\lim_{n\to\infty} \left\{ \int_0^\infty \frac{\mathrm{d}F_n(x)}{x+t} - \frac{1}{n} \operatorname{Tr} (C_n^*)^{-1} R_{n,t}^{(2)} C_n^{-1} \right\} = 0,$$

*where $R_{n,t}^{(2)} = \{R_{i,j,t}^{(2)}\}_{i,j=1,...,n}$, $R_{n,t}^{(1)} = \{R_{i,j,,t}^{(1)}\}_{i,j=1,...,n}$ and matrices $R_{n,t}^{(1)}, R_{n,t}^{(2)}$ satisfy the system of canonical equations $K_7$*

$$\begin{cases} R^{(1)}_{n,t} = [M_n^{-1} + \Theta^{(1)}_{n,t} + P_n^*(L_n(t) + \Theta^{(2)}_{n,t})^{-1}P_n]^{-1} \\ R^{(2)}_{n,t} = \left\{ L_n(t) + \Theta^{(2)}_{n,t} + P_n\left(M_n^{-1} + \Theta^{(1)}_{n,t}\right)^{-1} P_n^* \right\}^{-1}, \end{cases} \tag{13.4}$$

*where*

$$\Theta^{(1)}_{n,t} = \left\{ \frac{1}{n} \sum_{i=1,\dots,n} \sigma_{ij} R^{(2)}_{i,i,t}\delta_{jp} \right\}_{j,p=1,\dots,n}, \Theta^{(2)}_{n,t} = \left\{ \frac{1}{n} \sum_{i=1,\dots,n} \sigma_{ji} R^{(1)}_{i,i,t}\delta_{jp} \right\}_{j,p=1,\dots,n},$$

$L_n(t) = C_n^{-1}(I_n t + A_n)(C_n^*)^{-1}$, $P_n = C_n^{-1}B_nD_n^{-1}$ *and* $M_n = D_nD_n^*$. *The matrices* $R^{(1)}_{n,t}, R^{(2)}_{n,t}$ *are the solution of the system of canonical equations* $K_7$ *under* $\alpha = 1$

$$\begin{cases} K^{(1)}_{n,\alpha} = [\alpha M_n^{-1} + \Theta^{(1)}_{n,\alpha} + P_n^*(\alpha L_n(t) + \Theta^{(2)}_{n,\alpha})^{-1}P_n]^{-1}, \\ K^{(2)}_{n,\alpha} = \left\{ \alpha L_n(t) + \Theta^{(2)}_{n} + P_n\left(\alpha M_n^{-1} + \Theta^{(1)}_{n}\right)^{-1} P_n^* \right\}^{-1}, \end{cases} \tag{13.5}$$

*where*

$$\Theta^{(1)}_{n,\alpha} = \left\{ \frac{1}{n} \sum_{i=1,\dots,n} \sigma_{ij} K^{(2)}_{i,i,\alpha}\delta_{jp} \right\}_{j,p=1,\dots,n}, \Theta^{(2)}_{n} = \left\{ \frac{1}{n} \sum_{i=1,\dots,n} \sigma_{ji} K^{(1)}_{i,i,\alpha}\delta_{jp} \right\}_{j,p=1,\dots,n},$$

*and the entries* $K^{(1)}_{i,j,\alpha,t}, K^{(2)}_{i,j,\alpha,t}$ *of the matrices* $K^{(1)}_{n,\alpha,t}, K^{(2)}_{n,\alpha,t}$ *from the class of functions* $\Upsilon$.

*The entries* $K^{(2)}_{i,j,\alpha,t}$ *of the matrix* $K^{(2)}_{n,\alpha,t}$ *when* $\alpha = 1$ *are the G-transforms of some functions of bounded variation:*

$$[K^{(2)}_{i,j,\alpha,t}]_{\alpha=1} = \int_0^\infty \frac{\mathrm{d}G_{i,j}(x)}{t+x},$$

*where* $G_{i,j}(x), i \neq j$ *are certain functions of bounded variation and* $G_{i,i}(x)$ *are certain distribution functions.*

*There exists the unique solution* $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$ *of the system of canonical equations (13.5) in the set of analytical functions in* $\alpha > 0$

$$\Upsilon = \left\{ K^{(p)}_{i,j,\alpha,t} = \int_0^\infty \alpha(\alpha^2 + x)^{-1}\mathrm{d}F^{(p)}_{i,j,t}(x), i \geq j, i,j = 1,..,n, K^{(p)}_{n,\alpha,t} > 0_n, p = 1,2, \alpha > 0 \right\}, \tag{13.6}$$

*where* $K^{(1)}_{i,j,\alpha,t}, K^{(2)}_{i,j,\alpha,t}$ *are the entries of the non negative definite Hermitian matrices* $K^{(1)}_{n,\alpha,t} > 0_n, K^{(2)}_{n,\alpha,t} > 0_n$, *and* $F^{(1)}_{i,i,t}(x), F^{(2)}_{i,i,t}(x), i = 1,...,n$ *are the distribution functions and* $F^{(2)}_{i,j,t}(x), F^{(2)}_{i,j,t}(x), i \neq j$ *are the functions of bounded variation.*

**Remark 13.2.** *Note, that the name "Canonical equations for normalized spectral functions of random matrices" was introduced by V. L. Girko in [4] by analogy with the name of the canonical spectral representation of matrices or by analogy with the canonical systems which occupies a central position in the spectral theory of second order differential operators. Many canonical equations have been found since then. To maintain order, these canonical equations were numbered in [4]. Thus, the number of canonical equation published in this paper are* $K_1, K_7, K_{16}, K_{27}, K_{100}, ..., K_{106}$.

**Remark 13.3.** *We promised how the canonical equations (13.4) and (13.5) look like when the regularized parameter* $\epsilon > 0$ *of the matrices tends to zero(see remark 11.2). It is not difficult to do this and we give their form:*

$$\lim_{n\to\infty} \left\{ \int_0^\infty \frac{\mathrm{d}F_n(x)}{x+t} - \frac{1}{n}\mathrm{Tr}\, Q^{(2)}_{n,t} \right\} = 0,$$

*where* $Q^{(2)}_{n,t} = \{Q^{(2)}_{i,j,t}\}_{i,j=1,\dots,n}$, $Q^{(1)}_{n,t} = \{Q^{(1)}_{i,j,,t}\}_{i,j=1,\dots,n}$ *and matrices* $Q^{(1)}_{n,t}, Q^{(2)}_{n,t}$ *satisfy the system of canonical equations* $K_7$

$$\begin{cases} Q^{(1)}_{n,t} = D_n\left[I_n + D_n\Theta^{(1)}_{n,t}D_n^* + B_n^*\left(I_nt + A_n + C_n^*\Theta^{(2)}_{n,t}C_n\right)^{-1}B_n\right]^{-1}D_n^* \\ Q^{(2)}_{n,t} = \left\{I_nt + A_n + C_n^*\Theta^{(2)}_{n,t}C_n + B_n\left(I_n + D_n\Theta^{(1)}_{n,t}D_n^*\right)^{-1}B_n^*\right\}^{-1}, \end{cases}$$

$$\Theta^{(1)}_{n,t} = \left\{\frac{1}{n}\sum_{i=1,\dots,n}\sigma_{ij}\{C_nQ^{(2)}_nC_n^*\}_{ij}\delta_{jp}\right\}_{j,p=1,\dots,n}, \Theta^{(2)}_{n,t} = \left\{\frac{1}{n}\sum_{i=1,\dots,n}\sigma_{ji}Q^{(1)}_{i,i,t}\delta_{jp}\right\}_{j,p=1,\dots,n}.$$

*The matrices* $Q^{(1)}_{n,t}, Q^{(2)}_{n,t}$ *are the solution of the system of canonical equations* $K_7$ *under* $\alpha = 1$

$$\begin{cases} K^{(1)}_{n,\alpha} = D_n\left[\alpha I_n + D_n^*\Theta^{(1)}_{n,\alpha}D_n + B_n\left(\alpha(I_nt + A_n) + C_n^*\Theta^{(2)}_{n,\alpha}C_n\right)^{-1}B_n^*\right]^{-1}D_n^* \\ K^{(2)}_{n,\alpha} = \left\{\alpha\left(I_nt + A_n\right) + C_n^*\Theta^{(2)}_nC_n + B_n\left(\alpha I_n + D_n\Theta^{(1)}_nD_n^*\right)^{-1}B_n^*\right\}^{-1} \end{cases},$$

*where*

$$\Theta^{(1)}_{n,\alpha} = \left\{\frac{1}{n}\sum_{i=1,\dots,n}\sigma_{ij}\{C_nK^{(2)}_{n,\alpha}C_n^*\}_{ii}\delta_{jp}\right\}_{j,p=1,\dots,n}, \Theta^{(2)}_n = \left\{\frac{1}{n}\sum_{i=1,\dots,n}\sigma_{ji}K^{(1)}_{i,i,\alpha}\delta_{jp}\right\}_{j,p=1,\dots,n}.$$

*There exists the unique solution* $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$ *of the system of canonical equations in the set* $\Upsilon$ *of analytical entrees in* $\alpha > 0$.

We have obtained an expression for the $G$-transform $\int_0^\infty(x+t)^{-1}\mathrm{d}F_n(x)$ with a positive parameter $t > 0$. Such a transform is sometimes more convenient in some cases, although the inverse formula for it becomes more complicated. But we should always keep in mind that if there exists a limit $\lim_{n\to\infty}\int_0^\infty(x+t)^{-1}\mathrm{d}F_n(x) = f(t)$ for all $t > 0$, then the function $f(t)$ will be analytic for all $t > 0$ and we can continue it analytically and replace the parameter $t$ by the complex parameter $-z, \Im z > 0$ and obtain limit for the Stiltjes transform

$$\lim_{n\to\infty}\int_0^\infty (x-z)^{-1}\mathrm{d}F_n(x) = f(-z), \Im z > 0.$$

*Proof of Theorem* 13.1. The main steps of the proof of this theorem coincide with the corresponding steps of the proof of Theorem 9.3.1 [4]. Nevertheless, for this proof to be selfcontaining, we repeat briefly these steps.

# 14 The first step of the REFORM method of the proof of Theorem 13.1, Perturbation formulas for the resolvent of random matrices. Self-averaging of the resolvents of random matrices

The fist step of the REFORM method consists in the following preparation of the trace of the resolvent for any matrix $Q_{2n}$ with bounded singular eigenvalues in the formula (12.1)

$$\frac{1}{n}\operatorname{Tr} Q_{2n}R_{2n,\alpha}(t) = \frac{\partial}{\partial\beta}\frac{1}{n}\ln\det[\beta Q_{2n} + V_{2n,\alpha}(t)]_{\beta=0}, \tag{14.1}$$

as the sum of martingale-differences, where

$$Q_{2n} = \{q_{ij}\} = \begin{Bmatrix} 0 & 0 \\ 0 & \mathrm{i}C_n^{-1}(C_n^*)^{-1} \end{Bmatrix}, \Gamma_{2n,\alpha}(t) = \begin{Bmatrix} \mathrm{i}\alpha M_n^{-1} & P_n^* \\ P_n & \mathrm{i}\alpha L_n(t) \end{Bmatrix}, H_{2n} = (h_{ij}) = \begin{Bmatrix} 0 & \Xi_n^* \\ \Xi_n & 0 \end{Bmatrix},$$

$$R_{2n,\alpha}(t) = [V_{2n,\alpha}(t)]^{-1}, V_{2n,\alpha}(t) = \{v_{ij,\alpha}(t)\} = \Gamma_{2n,\alpha}(t) + H_{2n}, t > 0, \alpha > 0,$$

**Lemma 14.1.** [1] *If for any $n$ the entries of the matrix $\Xi_n$ are independent, the singular eigenvalues of the matrices $A_n, C_n, D_n$ are bounded from below and above by some positive constants, $A_n$ is the Hermitian matrix, matrix $B_n$ has bounded singular values, then for any matrix $G_{2n}$ with bounded singular eigenvalues for any $\delta > 0$ and any $\alpha \geq c > 0, t > 0$*

$$|n^{-1}\mathrm{Tr}\, Q_{2n}R_{2n,\alpha,t} - \mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}R_{2n,\alpha,t}|^{2+\delta} \leq cn^{-1-\delta/2}.$$

This statement was proved in 1975 in the book [1] and was repeated in many further publications[2,3,4].

*Proof.* Let $W_{2n}(\beta) = \{w_{ij}(\beta)\} = \beta Q_{2n} + V_{2n}$, and assume that $\vec{w}_k(\beta)$ is the $k$th vector column of the matrix $W_{2n}(\beta)$ without the component $w_{kk}(\beta)$, $\vec{u}_k(\beta)$ is the $k$th vector row of the matrix $W_{2n}(\beta)$ without the component $w_{kk}(\beta)$ and $W_{2n}^{(k)}(\beta)$ is he matrix obtained from the matrix $W_{2n}(\beta)$ by deletion of the entries $w_{kj}(\beta), w_{ik}(\beta), i, j = 1, ..., 2n$, $\vec{q}_k(\alpha)$ is the $k$th vector column of the matrix $Q_{2n}$ without the component $q_{kk}$, $\vec{v}_k(\alpha)$ is the $k$th vector column of the matrix $V_{2n}(\alpha)$ without the component $v_{kk}(\alpha)$, $Q_{2n}^{(k)}$ is he matrix obtained from the matrix $Q_{2n}$ by deletion of the entries $q_{kj}, q_{ik}, i, j = 1, ..., 2n$, $R_{2n}^{(k)}(\alpha)$ is he matrix obtained from the matrix $R_{2n}(\alpha)$ by deletion of the entries $v_{kj}(\alpha), v_{ik}(\alpha), i, j = 1, ..., 2n$.

Consider the sum of martingal-differences $n^{-1}\mathrm{Tr}\, R_{2n,\alpha}(t) = n^{-1}\sum_{k=1}^{2n}\gamma_k(\alpha, t)$ using (14.1), where

$$\begin{aligned} \gamma_k(\alpha, t) \quad &= (\mathbf{E}_{k-1} - \mathbf{E}_k)\frac{\partial}{\partial\beta}\frac{1}{n}\ln\det[W_{2n}(\beta)]_{\beta=0} \\ &= (\mathbf{E}_{k-1} - \mathbf{E}_k)\frac{\partial}{\partial\beta}\frac{1}{n}\ln\left\{\frac{\det[W_{2n}(\beta)]}{\det[W_{2n}^{(k)}(\beta)]}\right\}_{\beta=0} \\ &= (\mathbf{E}_{k-1} - \mathbf{E}_k)\frac{\partial}{\partial\beta}\frac{1}{n}\ln\left\{w_{kk}(\beta) - \vec{u}_k(\beta)[W_{2n}^{(k)}(\beta)]^{-1}\vec{w}_k(\beta)\right\}_{\beta=0} \\ &= (\mathbf{E}_{k-1} - \mathbf{E}_k)\frac{q_{kk} - 2\vec{q}_k^{\,*}R_{2n}^{(k)}\vec{v}_k + \vec{v}_k^{\,*}R_{2n}^{(k)}Q_{2n}^{(k)}R_{2n}^{(k)}\vec{v}_k}{v_{kk} - \vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k}, \end{aligned} \tag{14.2}$$

and $\mathbf{E}_k$ is the conditional expectation under fixed entries $\xi_{ij}^{(n)}, i \geq k, j \geq k$.

Since $|q_{kk}| \leq c < \infty$, $|v_{kk}| > c > 0$, $\vec{q}_k^{\,*}\vec{q}_k \leq c < \infty$, we have

$$\begin{aligned} &|\vec{q}_k^{\,*}R_{2n}^{(k)}\vec{v}_k| \leq c\sqrt{\vec{v}_k^{\,*}R_{2n}^{(k)}[R_{2n}^{(k)}]^*\vec{v}_k}, \Im v_{kk} - \Im\vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k > c\alpha + |\Im\vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k|, \\ &|\vec{v}_k^{\,*}R_{2n}^{(k)}Q_{2n}^{(k)}R_{2n}^{(k)}\vec{v}_k| \leq c|\vec{v}_k^{\,*}R_{2n}^{(k)}R_{2n}^{(k)*}\vec{v}_k| \leq c|\Im\vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k|. \end{aligned}$$

Therefore

$$\left|\frac{q_{kk} - 2\vec{q}_k^{\,*}R_{2n}^{(k)}\vec{v}_k + \vec{v}_k^{\,*}R_{2n}^{(k)}Q_{2n}^{(k)}R_{2n}^{(k)}\vec{v}_k}{v_{kk} - \vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k}\right| \leq c\frac{1 + |\Im\vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k| + |\Im\vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k|^{1/2}}{c + |\Im\vec{v}_k^{\,*}R_{2n}^{(k)}\vec{v}_k|} \leq c. \tag{14.3}$$

This is the key point of the REFORM method and it is underlies all studies of the limiting distribution of the spectral functions of random matrices. Then we can use the bounds on the moments of martingales. See: S. W. Dharmadhikari, V. Fabian and K. Jogdeo, Bounds on the Moments of Martingales, *Ann. Math. Statist.***39**(1968), no. 5, 1719–1723.

**Lemma 14.2.** *For any $\delta > 0$ and $n = 1, 2, ...$*

$$\mathbf{E}\left|\sum_{j=1,...,n} \gamma_j(\alpha, t)\right|^{2+\delta} \leq C_\delta n^{\delta/2} \sum_{j=1,...,n} \mathbf{E}\left|\gamma_j(\alpha, t)\right|^{2+\delta}, \tag{14.4}$$

*where $C_\delta = 8(1+\delta)\max(1, 2^{-1+\delta})$.*

Using (14.1)–(14.3) we obtain very important inequality for any $\delta > 0$ $\mathbf{E}\,|\gamma_j(\alpha, t)|^{2+\delta} \leq c < \infty$ and we complete the proof of Lemma 14.1. □

# 15 The second step of the REFORM method of the proof of Theorem 13.1. The invariance principle for resolvents of random matrices under the $G$-Lindeberg condition

This second step consists in use of the sequences of matrices $\Xi_n^{(0)} = \Xi_n, \Xi_n^{(j)}, j = 1, ..., n$ obtained from the matrix $\Xi_n$ by replace of the entries of its first $j$ columns and rows by independent random variables which independent of the entries $\xi_{pl}^{(n)}$ of the matrix $\Xi_n$ and are distributed by Normal laws $N(0, \mathbf{E}\,|\xi_{pl}^{(n)}|^2)$.

Let

$$Q_{2n} = \{q_{ij}\} = \begin{Bmatrix} 0 & 0 \\ 0 & \mathrm{i}C_n^{-1}(C_n^*)^{-1} \end{Bmatrix},$$

$$T_{2n,\alpha,t}(k) = [Y_{2n}(k,t,\alpha)]^{-1}, Y_{2n}(k) = \{y_{ij}(k)\} = \Gamma_{2n} + H_{2n}(k,t,\alpha), \alpha > 0, t > 0,$$

$$\Gamma_{2n} = \begin{Bmatrix} \mathrm{i}\alpha M_n^{-1} & P_n^* \\ P_n & \mathrm{i}\alpha L_n(t) \end{Bmatrix}, H_{2n}(k) = (h_{ij}(k)) = \begin{Bmatrix} 0 & [\Xi_n^{(k)}]^* \\ \Xi_n^{(k)} & 0 \end{Bmatrix}.$$

Then we consider the equality

$$\mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}T_{2n}(0) - \mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}T_{2n}(n) = \sum_{k=0}^{2n} n^{-1}\rho_k,$$

where $\rho_k = \mathbf{E}\,\mathrm{Tr}\, Q_{2n}T_{2n}(k-1) - \mathrm{Tr}\, Q_{2n}T_{2n}(k)$.

As in the numerous papers [1–4] since 1975 we prove

**Lemma 15.1.** [1] *Under the conditions of Theorem 13.1 for any $\alpha > 0, t > 0$*

$$\lim_{n\to\infty} |\mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}T_{2n}(0,\alpha,t) - \mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}T_{2n}(n,\alpha,t)| = 0.$$

*Proof.* Let $\Omega_{2n}(\beta, k) = \{\omega_{ij}(\beta, k)\} = \beta Q_{2n} + Y_{2n}(k, t, \alpha)$, and assumr that $\vec{\omega}_k(\beta)$ is the $k$th vector column of the matrix $\Omega_{2n}(\beta, k)$ without the component $\omega_{kk}(\beta)$, $\vec{a}_k(\beta)$ is the $k$th vector row of the matrix $\Omega_{2n}(\beta, k)$ without the component $\omega_{kk}(\beta, k)$ and $\Omega_{2n}^{(k)}(\beta, k)$ is he matrix obtained from the matrix $\Omega_{2n}(\beta, k)$ by deletion of the entries $\omega_{kj}(\beta), \omega_{ik}(\beta), i, j = 1, ..., 2n$. Let $T^{(k)}(k)$ be the matrix obtained from the matrix $T(k)$ by deletion of the entries $y_{kj}, y_{ik}, i, j = 1, ..., 2n$. We use te similar notations for the matrices $Q_{2n}$ and $Y_{2n}$.

Then using (11.1) we have

$$\begin{aligned} \rho_k \quad &= \mathbf{E}\,\frac{\partial}{\partial\beta} \ln\det\left\{\beta q_{kk} + y_{kk}(k-1) - [\beta\vec{q}_k + \vec{y}_k(k-1)]^{\,*}(\Omega^{(k)}(\beta,k))^{-1}[\beta\vec{q}_k + \vec{y}_k(k-1)]\right\}_{\beta=0} \\ &-\mathbf{E}\,\frac{\partial}{\partial\beta} \ln\det\left\{\beta q_{kk} + y_{kk}(k) - [\beta\vec{q}_k + \vec{y}_k(k)]^{\,*}(\Omega^{(k)}(\beta,k))^{-1}[\beta\vec{q}_k + \vec{y}_k(k)]\right\}_{\beta=0}. \end{aligned} \tag{15.1}$$

As in the previous section we have

$$\begin{aligned}\rho_k(t) \qquad &= \mathbf{E}\,\frac{q_{kk} - 2\vec{q}_k^{\,*}T^{(k)}(k-1)\vec{y}_k(k-1) + \vec{y}_k^{\,*}(k-1)T^{(k)}(k-1)Q_{2n}^{(k)}T^{(k)}(k-1)\vec{y}_k(k-1)}{y_{kk}(k-1) - \vec{y}_k^{\,*}(k-1)T^{(k)}(k-1)\vec{y}_k(k-1)}\\ &-\mathbf{E}\,\frac{q_{kk} - 2\vec{q}_k^{\,*}T^{(k)}(k)\vec{y}_k(k) + \vec{y}_k^{\,*}(k)T^{(k)}(k)Q_{2n}^{(k)}T^{(k)}(k)\vec{y}_k(k)}{y_{kk}(k) - \vec{y}_k^{\,*}(k)T^{(k)}(k)\vec{y}_k(k)}. \end{aligned} \tag{15.2}$$

Since, as follows from the previous section, these expressions are bounded and using the generalized Lindeberg condition for (15.1) and (15.2) as in [1–5] we obtain that

$$\lim_{n\to\infty} |\mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}T_{2n}(0,\alpha,t) - \mathbf{E}\, n^{-1}\mathrm{Tr}\, Q_{2n}T_{2n}(n,\alpha,t)| = 0.$$

□

This proof is uncomplicated, but cumbersome. Note that for any vector $\vec{c}_n$ of unit length we have $\mathbf{E}\,|\vec{c}_n^{\,*}\vec{\xi}_k|^2 \le \epsilon_n, k = 1, ..., n$ and also for any Hermitian matrix $C_n = \{c_{ij}\}$ with bounded eigenvalues

$$\mathbf{E}\,|\vec{\xi}_n^{\,*}C_n\vec{\xi}_k - \sum_{i=1,..,n} c_{ii}\xi_{ik}^2|^2 \le \epsilon_n, k = 1, ..., n, \lim_{n\to\infty} \epsilon_n = 0.$$

Therefore

$$|\rho_k(t)| \le c\mathbf{E}\left|\sum_{i=1,..,m_n} c_{ii}[\xi_{ik}^2 - \sigma_{ik}]\right| + \mathbf{E}\left|\sum_{i=1,..,n} d_{ii}[\xi_{ki}^2 - \sigma_{ki}]\right| + \epsilon_n,$$

where $|c_{ii}|$ and $|d_{ii}|$ are some bounded complex numbers. Then using the Generalized Lindeberg condition we complete the proof of Lemma 15.1.

# 16 The third step of the REFORM method of the proof of Theorem 13.1. The resolvent equality of random matrices. The canonical equations $K_1$ and $K_7$

The facts presented in this section are very important not only for Gram random matrices but also for nonsymmetric random matrices. Therefore, we can consider the following theorem as a bridge between the theories of Hermitian and non-Hermitian random matrices.

Denote(see (12.1))

$$R_{2n}(t,\alpha) = \begin{Bmatrix} R_{n,\alpha}^{(1)} & R_{n,\alpha}^{(3)} \\ R_{n,\alpha}^{(4)} & R_{n,\alpha}^{(2)} \end{Bmatrix},$$

where $R_{i,j,\alpha}$ are the entries of the matrix $R_{2n}(t,\alpha) = \{R_{i,j,\alpha}\} = [H_{2n} + \Gamma_{2n}]^{-1}$, $R_{n,\alpha}^{(1)} = \{R_{i,j,\alpha}^{(1)}\}_{i,j=1,...,n}$, $R_{n,\alpha}^{(2)} = \{R_{i,j,\alpha}^{(2)}\}_{i,j=1,...,n}$. Remember that we already have replaced the entries of the matrix $H_{2n}$ by normally distributed random variables.

**Theorem 16.1.** *Under the conditions of Theorem 13.1 the matrices $\mathbf{E}\,R_{n,\alpha}^{(1)}, \mathbf{E}\,R_{n,\alpha}^{(2)}$ satisfy the system of canonical equations $K_7$*

$$\begin{cases} \mathbf{E}\,R_{n,\alpha}^{(1)} = [\mathrm{i}\alpha M_n^{-1} - \Theta_{n,\alpha}^{(1)} - P_n^*(\mathrm{i}\alpha L_n(t) - \Theta_{n,\alpha}^{(2)})^{-1}P_n]^{-1} + E_n \\ \mathbf{E}\,R_{n,\alpha}^{(2)} = \left\{\mathrm{i}\alpha L_n(t) - \Theta_{n,\alpha}^{(2)} - P_n\left(\mathrm{i}\alpha M_n^{-1} - \Theta_{n,\alpha}^{(1)}\right)^{-1} P_n^*\right\}^{-1} + E_n, \end{cases} \tag{16.1}$$

*where the entries $\epsilon_{ij}$ of the non random matrix $E_n = \{\epsilon_{ij}\}$ satisfy the equalities for any matrices $Q_n$ with bounded singular values*

$$\lim_{n\to\infty} |\operatorname{Tr} Q_n E_n| = 0,\ \lim_{n\to\infty} \max_{i,j=1,\dots,n} |\epsilon_{ij}| = 0,$$

$$\Theta^{(1)}_{n,\alpha} = \left\{\frac{1}{n}\sum_{i=1,\dots,n} \sigma_{ij}\mathbf{E}\, R^{(2)}_{i,i,\alpha}\delta_{jp}\right\}_{j,p=1,\dots,n},\ \Theta^{(2)}_{n,\alpha} = \left\{\frac{1}{n}\sum_{i=1,\dots,n} \sigma_{ji}\mathbf{E}\, R^{(1)}_{i,i,\alpha}\delta_{jp}\right\}_{j,p=1,\dots,n},$$

$L_n(t) = C_n^{-1}(I_n t + A_n)(C_n^*)^{-1}, P_n = C_n^{-1}B_n D_n^{-1}$ *and* $M_n = D_n D_n^*$.

*Proof.* Denote

$$G_{2n} = \begin{Bmatrix} 0 & 0 \\ 0 & \mathrm{i}C_n^{-1}(C_n^*)^{-1}\end{Bmatrix}\left[\begin{Bmatrix} \mathrm{i}\alpha M_n^{-1} & P_n^* \\ P_n & \mathrm{i}\alpha L_n(t)\end{Bmatrix} + \begin{Bmatrix} -\Theta^{(1)}_{n,\alpha} & 0 \\ 0 & -\Theta^{(2)}_n\end{Bmatrix}\right]^{-1} = Q_{2n}T_{2n,\alpha}, \tag{16.2}$$

where

$$\begin{aligned}
Q_{2n} &= \begin{Bmatrix} 0 & 0 \\ 0 & \mathrm{i}C_n^{-1}(C_n^*)^{-1}\end{Bmatrix}, T_{2n,\alpha} = [\Gamma_{2n} - \Theta_{2n,\alpha}]^{-1}, \Theta_{2n,\alpha} = \begin{Bmatrix} \Theta^{(1)}_{n,\alpha} & 0 \\ 0 & \Theta^{(2)}_{n,\alpha}\end{Bmatrix},\\
R_{2n,\alpha} &= \Big[\Gamma_{2n} + H_{2n}\Big]^{-1}, t>0, \alpha>0, \Gamma_{2n} = \begin{Bmatrix} \mathrm{i}\alpha M_n^{-1} & P_n^* \\ P_n & \mathrm{i}\alpha L_n(t)\end{Bmatrix}, H_{2n} = (h_{ij}) = \begin{Bmatrix} 0 & \Xi_n^* \\ \Xi_n & 0\end{Bmatrix},\\
H^{(k)}_{2n} &= \{h_{pl}\delta_{kp} + h_{pl}\delta_{lk}\}_{p,l=1,\dots,2n} = \vec{h}_k\vec{e}_k^{\,\mathrm{T}} + \vec{e}_k\vec{h}_k^{\,\mathrm{T}}, R^{(k)}_{2n,\alpha} = [H_{2n} + \Gamma_{2n} - H^{(k)}_{2n}]^{-1},\\
R_{2n,\alpha} &- R^{(k)}_{2n,\alpha} = -R^{(k)}_{2n,\alpha}[H^{(k)}_{2n}]R_{2n,\alpha},
\end{aligned} \tag{16.3}$$

$\vec{h}_k$ is the $k$-th column vector of the matrix $H_{2n}$, $\vec{e}_k^{\,\mathrm{T}} = \{\delta_{ik}, i = 1,\dots,n\}$ is the $k$-th unique vector. The entries of the $k$-th column and $k$-th row of the matrix $H^{(k)}_{2n}$ are equal to the corresponding entries of the matrix $H_{2n}$, and the other entries of the matrix $H^{(k)}_{2n}$ are equal to zero.

We follow the derivation of the equation $K_{16}$ from the book [4]. By virtue of the invariance principle we will assume that all entries of the matrix $\Xi_n$ are distributed according to the normal law. Since we have proved the self-averaging of the traces of random matrices we can very simplify the proof of our theorem and now we consider the following representations for the resolvent. Using (16.3) and any non random matrix $\Omega_{2n}$ with bounded singular values we have

$$\begin{aligned}
\frac{1}{n}\mathbf{E}\,[\operatorname{Tr}\Omega_{2n}T_{2n,\alpha} &- \operatorname{Tr}\Omega_{2n}R_{2n,\alpha}] = \frac{1}{n}\mathbf{E}\,[\operatorname{Tr}\Omega_{2n}T_{2n,\alpha}(H_{2n}+\Theta_{2n})R_{2n,\alpha}]\\
&= \frac{1}{n}\mathbf{E}\operatorname{Tr}\{H_{2n}R_{2n,\alpha}\Omega_{2n}T_{2n,\alpha} + \Theta_{2n}R_{2n,\alpha}\Omega_{2n}T_{2n}\}\\
&= \frac{1}{n}\mathbf{E}\,[\operatorname{Tr}\{-H_{2n}R^{(k)}_{2n,\alpha}H^{(k)}_{2n}R_{2n,\alpha}\Omega_{2n}T_{2n,\alpha} + H^{(k)}_{2n}R^{(k)}_{2n,\alpha}\Omega_{2n}T_{2n,\alpha} + \Theta_{2n}R_{2n,\alpha}\Omega_{2n}T_{2n}\}\\
&= -\frac{1}{n}\mathbf{E}\sum_{k=1,\dots,2n}\left[\left(\vec{h}_k^{\,\mathrm{T}}R^{(k)}_{2n}\vec{h}_k\vec{e}_k^{\,\mathrm{T}} + \vec{h}_k^{\,\mathrm{T}}R^{(k)}_{2n}\vec{e}_k\vec{h}_k^{\,\mathrm{T}}\right)R_{2n}\Omega_{2n}T_{2n}\right]_{kk}\\
&+\frac{1}{n}\mathbf{E}\sum_{k=1,\dots,2n}\left[\vec{h}_k^{\,\mathrm{T}}R^{(k)}_{2n}\Omega_{2n}T_{2n}\right]_{kk} + \frac{1}{n}\sum_{k=1,\dots,2n}\Big(\Theta_{2n}R_{2n,\alpha}\Omega_{2n}T_{2n}\Big)_{kk},
\end{aligned} \tag{16.4}$$

where $\vec{h}_k$ is the $k$th column vector of the matrix $H_{2n}$.

Obviously $\mathbf{E}\sum_{k=1,\dots,2n}\left[\vec{h}_k^{\,\mathrm{T}}R^{(k)}_{2n}\Omega_{2n}T_{2n}\right]_{kk} = 0$ and since we already have replaced the entries of the matrix $H_{2n}$ by normally distributed random variables and using once again the self-averaging of the quadratics forms of the resolvent of random matrices $\vec{h}_k^{\,\mathrm{T}}R^{(k)}_{2n}\vec{h}_k$ we get

$$\lim_{n\to\infty}\max_k \mathbf{E}\left|\vec{h}_k^{\mathrm{T}} R_{2n}^{(k)}\vec{h}_k - \mathbf{E}\,\vec{h}_k^{\mathrm{T}}\mathbf{E}\,R_{2n}\vec{h}_k\right|^2 = 0, \tag{16.5}$$

$$\lim_{n\to\infty}\max_k \mathbf{E}\left|\left[\vec{h}_k^{\mathrm{T}} R_{2n}^{(k)}\vec{e}_k\vec{h}_k^{\mathrm{T}} R_{2n}\Omega_{2n}T_{2n}\right]_{kk}\right|^2 \le c\lim_{n\to\infty}\max_k \mathbf{E}\left|\vec{h}_k^{\mathrm{T}} R_{2n}^{(k)}\vec{e}_k\right|^2 \mathbf{E}\,\vec{h}_k^{\mathrm{T}}\vec{h}_k = 0. \tag{16.6}$$

Therefore

$$\frac{1}{n}\mathbf{E}\sum_{k=1,\dots,2n}\left[\left(\vec{h}_k^{\mathrm{T}} R_{2n}^{(k)}\vec{h}_k\vec{e}_k^{\mathrm{T}} + \vec{h}_k^{\mathrm{T}} R_{2n}^{(k)}\vec{e}_k\vec{h}_k^{\mathrm{T}}\right)R_{2n}\Omega_{2n}T_{2n}\right]_{kk} = \frac{1}{n}\sum_{k=1,\dots,2n}(\Theta_{2n}\mathbf{E}\,R_{2n,\alpha}\Omega_{2n}T_{2n})_{kk} + \epsilon_n,$$

where $\lim_{n\to\infty}\epsilon_n = 0$. Then using (16.4)–(16.6) we get

$$\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\,[\mathrm{Tr}\,\Omega_{2n}T_{2n,\alpha} - \mathrm{Tr}\,\Omega_{2n}R_{2n,\alpha}] = 0.$$

Since the entries of the matrix $\Omega_{2n}$ are arbitrary we obtain for the blocks of the matrix

$$\begin{Bmatrix}\mathbf{E}\,R_{n,\alpha}^{(1)} & \mathbf{E}\,R_{n,\alpha}^{(3)}\\ \mathbf{E}\,R_{n,\alpha}^{(4)} & \mathbf{E}\,R_{n,\alpha}^{(2)}\end{Bmatrix} = T_{2n,\alpha} + E_n = [\Gamma_{2n} - \Theta_{2n,\alpha}]^{-1} + E_n, \tag{16.7}$$

$$\mathbf{E}\,R_{n,\alpha}^{(1)} = [\mathrm{i}\alpha M_n^{-1} - \Theta_{n,\alpha}^{(1)} - P_n^*(\mathrm{i}\alpha L_n(t) - \Theta_{n,\alpha}^{(2)})^{-1}P_n]^{-1} + E_n,$$

$$\mathbf{E}\,R_{n,\alpha}^{(2)} = \left\{\mathrm{i}\alpha L_n(t) - \Theta_n^{(2)} - P_n\left(\mathrm{i}\alpha M_n^{-1} - \Theta_n^{(1)}\right)^{-1} P_n^*\right\}^{-1} + E_n.$$

Equation (16.7) is called the canonical equation $K_1$.

Then we get

$$\frac{1}{n}\mathrm{Tr}\,Q_{2n}T_{2n,\alpha} = \frac{1}{n}\mathrm{Tr}\,C_n^{-1}\mathbf{E}\,R_{n,\alpha}^{(2)}(C_n^*)^{-1} + \epsilon_n.$$

□

## 17 The forth step. The $G$-Matrix Expansion Method

However, we still have an equation for the entries of the matrices $\mathbf{E}\,R_{n,\alpha}^{(1)}, \mathbf{E}\,R_{n,\alpha}^{(2)}$ with some error $\epsilon_n$ tending to zero when $n$ tends to infinity. For the first time, it was proved in [2–5] that it is possible to find an equation without this error that will approximate the matrices $\mathbf{E}\,R_{n,\alpha}^{(1)}, \mathbf{E}\,R_{n,\alpha}^{(2)}$ well. We called the procedure for finding this equation *the Matrix Expansion Method*, the idea of which is quite simple. This method is one of the most important methods and was established in [4]. Consider the block matrices for any $s = 1, 2, \dots$ and fixed $n$

$$Z_{2ns\times 2ns}(\alpha) = \{\mathrm{i}\alpha(\Gamma_{2n})_{\delta_{pl}} + P_{2ns\times 2ns} + s^{-1/2}\Phi_{2n}^{(p,l)}\}_{p,l=1,..,s}, P_{2ns\times 2ns} = \left\{\begin{Bmatrix}0 & P_n^*\\ P_n & 0\end{Bmatrix}\delta_{pl}\right\}_{p,l=1,\dots,s}$$

where $\Phi_{2n}^{(p,l)} = \Phi_{2n}^{(l,p)}, l, p = 1, \dots, s$, the matrices $\Phi_{2n}^{(p,l)} = \{\phi_{ij}^{(p,l)}\}_{i,j=1,\dots,2n}, p \ge l, l, p = 1, \dots, s$ are independent with independent entries $\phi_{ij}^{(p,l)}$ distributed by normal law and $\mathbf{E}\,\phi_{ij}^{(p,l)} = 0, \mathbf{E}\,[\phi_{ij}^{(p,l)}]^2 = \mathbf{E}\,[h_{ij}]^2, i, j = 1, \dots, 2n, p \ge l, l, p = 1, \dots, s$.

Denote the set of analytical functions in $\alpha \ne 0$

$$\Upsilon = \left\{K_{i,j,\alpha}^{(p)} :\quad K_{i,j,\alpha}^{(p)} = \int_{-\infty}^{\infty}(\mathrm{i}\alpha + x)^{-1}\mathrm{d}F_{i,j,\alpha}^{(p)}(x), i \ge j, p = 1, 2\right\}, \tag{17.1}$$

where $K^{(1)}_{i,j,\alpha}, K^{(2)}_{i,j,\alpha}$ are the entries of the matrices $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$, $F^{(1)}_{i,i,\alpha}(x), F^{(2)}_{i,i,\alpha}(x)$ are the distribution functions and $F^{(2)}_{i,j,\alpha}(x), F^{(2)}_{i,j,\alpha}(x)$ are the functions of bounded variation.

Then we have

**Lemma 17.1.** *Under the conditions of Theorem 13.1 there exists the unique solution $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$ of the system of canonical equations*

$$\begin{cases} K^{(1)}_{n,\alpha} = [\mathrm{i}\alpha M_n^{-1} - \Theta^{(1)}_{n,\alpha} - P_n^*(\mathrm{i}\alpha L_n(t) - \Theta^{(2)}_{n,\alpha})^{-1} P_n]^{-1}, \\ K^{(2)}_{n,\alpha} = \left\{ \mathrm{i}\alpha L_n(t) - \Theta^{(2)}_n - P_n \left( \mathrm{i}\alpha M_n^{-1} - \Theta^{(1)}_n \right)^{-1} P_n^* \right\}^{-1}, \end{cases} \tag{17.2}$$

*in the set $\Upsilon$ of analytical functions in $\alpha \neq 0$, where*

$$\Theta^{(1)}_{n,\alpha} = \left\{ \frac{1}{n} \sum_{i=1,\dots,n} \sigma_{ij} K^{(2)}_{i,i,\alpha} \delta_{jp} \right\}_{j,p=1,\dots,n}, \Theta^{(2)}_n = \left\{ \frac{1}{n} \sum_{i=1,\dots,n} \sigma_{ji} K^{(1)}_{i,i,\alpha} \delta_{jp} \right\}_{j,p=1,\dots,n},$$

*and the entries $K^{(1)}_{i,j,\alpha}, K^{(2)}_{i,j,\alpha}$ of the matrices $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$ are the Stiltjes transform of some functions of bounded variation $F^{(1)}_{i,j,t}(x), F^{(2)}_{i,j,t}(x)$:*

$$K^{(p)}_{i,j,\alpha,t} = \int_{-\infty}^{\infty} (\mathrm{i}\alpha + x)^{-1} \mathrm{d}F^{(p)}_{i,j,t}(x), i \geq j, p = 1, 2,$$

*$F^{(1)}_{i,i,t}(x), F^{(2)}_{i,i,t}(x)$ are the distribution functions. The entries , $K^{(2)}_{i,j,\alpha}$ of the matrix $K^{(2)}_{n,\alpha}$ when $\alpha = 1$ are the $G$-transform of some functions of bounded variation:*

$$[K^{(2)}_{i,j,\alpha}]_{\alpha=1} = -\mathrm{i} \int_0^{\infty} (t + x)^{-1} \mathrm{d}G_{i,j}(x), i \geq j.$$

*Proof.* We consider the normalized $G$-transform for block matrices $Z_{2ns\times 2ns}(\alpha)$ and any block matrix $\Omega_{2ns\times 2ns} = \{\Omega_{2n}\delta_{ij}\}_{i,j=1,\dots,n}$ with bounded singular eigenvalues:

$$f_n(\alpha, s) = \frac{1}{sn} \mathrm{Tr}\, \Omega_{2ns\times 2ns} Z^{-1}_{2ns\times 2ns}(\alpha).$$

Now the parameter $n$ is fixed and the limit is considered already when $s$ tends to infinity.

Let $\Psi^{(s)}_{2n\times 2n} = (ns)^{-1} \sum_{i=1,\dots,s} \mathbf{E}\, \{Z^{-1}_{2ns\times 2ns}\}_{ii}$, where $\{Z^{-1}_{2ns\times 2ns}\}_{ii}$ are diagonal blocks of the matrix $Z^{-1}_{2ns\times 2ns}$.

Repeating the proof of the Theorem 16.1 since the entries of the matrix $\Omega_{2n}$ are arbitrary, we get

$$\frac{1}{sn} \mathbf{E}\, \mathrm{Tr}\, \Omega_{2ns\times 2ns} Z^{-1}_{2ns\times 2ns}(\alpha) = \frac{1}{n} \mathrm{Tr}\, \Omega_{2n\times 2n} \Pi_{2n\times 2n}(s, \alpha) + \epsilon_s,$$

where $\lim_{s\to\infty} \epsilon_s = 0$, $\Pi_{2n\times 2n}(s, \alpha) = (ns)^{-1} \sum_{i=1,\dots,s} \mathbf{E}\, \{Z^{-1}_{2ns\times 2ns}\}_{ii}$, $\{Z^{-1}_{2ns\times 2ns}\}_{ii}$ are diagonal blocks of the matrix $Z^{-1}_{2ns\times 2ns}$ and

$$\Pi_{2n\times 2n}(s, \alpha) = \begin{Bmatrix} \Pi^{(1)}_{n,s,\alpha} & \Pi^{(3)}_{n,s,\alpha} \\ \Pi^{(4)}_{n,s,\alpha} & \Pi^{(2)}_{n,s,\alpha} \end{Bmatrix},$$

$$\Pi^{(1)}_{n,\alpha}(s, \alpha) = [\mathrm{i}\alpha M_n^{-1} - \Theta^{(1)}_{n,s,\alpha} - P_n^*(\mathrm{i}\alpha L_n(t) - \Theta^{(2)}_{n,s,\alpha})^{-1} P_n]^{-1} + V^{(s)}_{2n\times 2n},$$

$$\Pi^{(2)}_{n,s,\alpha} = \left\{ \mathrm{i}\alpha L_n(t) - \Theta^{(2)}_n - P_n \left( \mathrm{i}\alpha M_n^{-1} - \Theta^{(1)}_n \right)^{-1} P_n^* \right\}^{-1} C_n^{-1} + V^{(s)}_{2n\times 2n},$$

$$\Theta^{(1)}_{n,s,\alpha} - \mathrm{i}\left\{\frac{1}{n}\sum_{i=1,\ldots,n}\sigma_{ij}\Pi^{(2)}_{i,i,s,\alpha}\delta_{jp}\right\}_{j,p=1,\ldots,n}, \Theta^{(2)}_{n,s,\alpha} = \left\{\frac{1}{n}\sum_{i=1,\ldots,n}\sigma_{ji}\Pi^{(1)}_{i,i,s,\alpha}\delta_{jp}\right\}_{j,p=1,\ldots,n},$$

and for any matrix $\Omega_{2n\times 2n}$ with bounded singular values $\lim_{s\to\infty} n^{-1}\mathrm{Tr}\,\Omega_{2n\times 2n}V^{(s)}_{2n\times 2n} = 0$.

Obviously $[Z_{2ns\times 2ns}(\alpha)]^{-1} = T_{2ns\times 2ns}\left\{\mathrm{i}\alpha I_{2ns\times 2ns}+T_{2ns\times 2ns}(P_{2ns\times 2ns}+[s^{-1/2}(\Phi^{(p,l)}_{2n})_{p,l=1,..,s}])T_{2ns\times 2ns}\right\}^{-1} T_{2ns\times 2ns}$, where

$$T_{2ns\times 2ns} = \left\{\begin{Bmatrix} M_n^{1/2} & 0 \\ 0 & L_n^{(-1/2)}(t)\end{Bmatrix}\delta_{pl}\right\}_{p,l=1,\ldots,s}, P_{2ns\times 2ns} = \left\{\begin{Bmatrix} 0 & P_n^* \\ P_n & 0\end{Bmatrix}\delta_{pl}\right\}_{p,l=1,\ldots,s}.$$

Then for all $p,l = 1,...,n$

$$\mathbf{E}\{\Psi^{(s)}_{2n\times 2n}\}_{pl} = (ns)^{-1}\sum_{i=1,\ldots,s}\left[\mathbf{E}\{Z^{-1}_{2ns\times 2ns}\}_{ii}\right]_{pl} = (ns)^{-1}\sum_{k=1,\ldots,2ns}\mathbf{E}\,\frac{c_{k,p,l,n,s}}{\mathrm{i}\alpha+\lambda_{k,n,s}},$$

where $\lambda_{k,n,s}$ are the eigenvalues of the matrix

$$M_{2ns\times 2ns} = T_{2ns\times 2ns}[P_{2ns\times 2ns} + s^{-1/2}(\Phi^{(p,l)}_{2n})_{p,l=1,..,s}]T_{2ns\times 2ns}$$

and $c_{k,p,l,n,s}, |c_{k,p,l,n,s}|^2 \le c$ are some random bounded variables.

The entries of the matrix $\Psi^{(s)}_{2n\times 2n}$ are the $G$-transforms of some functions $F^{(s)}_{ij}(x), i,j = 1,...,2n$ of bounded variation. Therefore, we can choose weakly convergent subsequences of this finite number of functions $F^{(s')}_{ij}(x), i,j = 1,...,2n, s' \to \infty$ (under fixed $n$)to a functions $F_{ij}(x), i,j = 1,...,2n$ of bounded variation since for all $p,l = 1,...,n$ and fixed $n$

$$\lim_{h\to\infty}\lim_{s\to\infty}\frac{1}{ns}\sum_{k=1,\ldots,2ns}\mathbf{E}\,\frac{|c_{k,p,l,n,s}|}{|\mathrm{i}\alpha+\lambda_{k,n,s}|}\chi\{|\lambda_{k,n,s}|\ge h\} \le \lim_{h\to\infty}\lim_{s\to\infty}\frac{c}{\sqrt{\alpha^2+h^2}}\mathbf{E}\,\mathrm{Tr}\,M_{2ns\times 2ns}M^*_{2ns\times 2ns} = 0,$$

and

$$\lim_{h\to\infty}\lim_{s\to\infty}\frac{1}{ns}\sum_{k=1,\ldots,2ns}\mathbf{E}\,\chi\{|\lambda_{k,n,s}|\ge h\} \le \lim_{h\to\infty}\lim_{s\to\infty}(nsh^2)^{-1}\mathbf{E}\,\mathrm{Tr}\,M_{2ns\times 2ns}M^*_{2ns\times 2ns} = 0.$$

Then there will be limits of these $G$-transforms under $s' \to \infty$ and they will satisfy the system of the equations (17.2). Let us prove the uniqueness of the solution of these equations (17.2) in the class of analytic functions (17.1). Suppose the contrary and let there exist two solutions $\Pi_{2n\times 2n}(s,\alpha), \Omega_{2n\times 2n}(s,\alpha)$ from the class $\Upsilon$. Then the inequality follows from the equations (17.2)

$$\max_{i,j=1,\ldots,2n}|\Pi_{ij}(s,\alpha)-\Omega_{ij}(s,\alpha)| \le c\alpha^{-2}\max_{i,j=1,\ldots,2n}|\Pi_{ij}(s,\alpha)-\Omega_{ij}(s,\alpha)|$$

and therefore these solutions coincide under $c|\alpha|^{-2} < 1$ and so, by virtue of their analyticity, they will coincide for all $\alpha \ne 0$. But then the existence of limit follows

$$\lim_{s\to\infty}\Pi_{2n\times 2n}(s,\alpha) = K_{2n\times 2n,\alpha}, \alpha \ne 0$$

and $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$ will satisfy the system of equations (17.2).

Let us prove that the entries $K^{(1)}_{i,j,\alpha}, K^{(2)}_{i,j,\alpha}$ of the matrices $K^{(1)}_{n,\alpha}, K^{(2)}_{n,\alpha}$ when $\alpha = 1$ are the Stiltjes transform of some functions of bounded variation $F^{(1)}_{i,j}(x), F^{(2)}_{i,j}(x)$:

Consider $\Psi_{2n\times 2n} = \begin{Bmatrix} \Psi_{n,s}^{(1)} & \Psi_{n,s,}^{(3)} \\ \Psi_{n,s,}^{(4)} & \Psi_{n,s,}^{(2)} \end{Bmatrix} = (ns)^{-1}\sum_{i=1,\dots,s} \mathbf{E}\{Z_{2ns\times 2ns}^{-1}\}_{ii}$, where $\{Z_{2ns\times 2ns}^{-1}\}_{ii}$ are diagonal blocks of the matrix $Z_{2ns\times 2ns}^{-1}$,

Obviously,

$$[K_{i,j,\alpha}^{(2)}]_{\alpha=1} = -\mathrm{i}\sum_{k=1,\dots,n} \mathbf{E}\,\frac{d_{ijk}}{(t+u_k)},$$

where $u_k \geq 0$ and $d_{ij}$ are certain bounded numbers. Therefore, repeating the previous proof we complete the statement of the Lemma 17.1.

□

# 18 The fifth step. Approximation by canonical equation $K_7$

So, we have obtained two systems of canonical equations (16.1) and (17.2). Recall that we do not need the matrix itself $R_{n,\alpha}^{(2)}$, but its normalized trace $\frac{1}{n}\mathrm{Tr}\,(C_n^*)^{-1}R_{n,\alpha}^{(2)}C_n^{-1}$. Therefore, using these equations (16.1) and (17.2) we obtain for any $\alpha > 0$ and $p = 1, 2$

$$\max_{i,j=1,\dots,n,p=1,2} |\mathbf{E}\,R_{ij}^{(p)}(\alpha) - K_{ij}^{(p)}(\alpha)| \leq c\alpha^{-2} \max_{i,j=1,\dots,n,p=1,2} |\mathbf{E}\,R_{ij}^{(p)}(\alpha) - K_{ij}^{(p)}(\alpha)| + \epsilon_n.$$

Therefore, if $c\alpha^{-2} < 1$ then

$$\lim_{n\to\infty} \max_{i,j=1,\dots,n,p=1,2} |\mathbf{E}\,R_{ij}^{(p)}(\alpha) - K_{ij}^{(p)}(\alpha)| = 0$$

and hence if $c\alpha^{-2} < 1$ then

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}\,(C_n^*)^{-1}[\mathbf{E}\,R_{n,\alpha}^{(2)} - K_{n,\alpha}^{(2)}]C_n^{-1} = 0. \tag{18.1}$$

Obviously, the functions $\mathbf{E}\,R_{ij}^{(p)}(\alpha), K_{ij}^{(p)}(\alpha)$ and $\frac{1}{n}\mathrm{Tr}\,(C_n^*)^{-1}[\mathbf{E}\,R_{n,\alpha}^{(2)} - K_{n,\alpha}^{(2)}]C_n^{-1}$ are $G$-transforms of some functions of bounded variation. Therefore, for any convergent subsequence of these functions, subsequences of $G$-transforms will also converge to $G$-transforms of some functions of bounded variation. Then reasoning from the contrary choosing convergent subsequences due to the fact that the Stiltjes limit transforms are analytical functions in $\alpha > 0$ the limit (18.1) will be valid for any $\alpha > 0$. Then tending the parameter $\alpha$ in the equation (18.1) to one we get

$$\lim_{\alpha\to 1}\lim_{n\to\infty} \left\{\frac{1}{n}\mathrm{Tr}\,Q_{2n}\mathbf{E}\,R_{2n} - \frac{1}{n}\mathrm{Tr}\,(C_n^*)^{-1}K_{n,\alpha}^{(2)}C_n^{-1}\right\} = 0.$$

Now we can get rid of the imaginary parameter i in the system of equations (16.1) and (17.2) by overidentifying $R_{n,\alpha}^{(1)} := \mathrm{i}\mathbf{E}\,R_{n,\alpha}^{(1)}, R_{n,\alpha}^{(2)} := \mathrm{i}\mathbf{E}\,R_{n,\alpha}^{(2)}$, $K_{n,\alpha}^{(1)} := \mathrm{i}K_{n,\alpha}^{(1)}, K_{n,\alpha}^{(2)} := \mathrm{i}K_{n,\alpha}^{(2)}$ and then obtain the main equations of the Theorem 13.1. So we have proved the main statement of the Theorem 13.1.

# 19 The generalization of the canonical equations $K_1$ and $K_7$ for the sum of independent random matrices $n^{-1}\sum_{j=1,\dots,n}\Xi_n^{(j)}$. Canonical equations $K_{27}$ and $K_{28}$

A natural generalization of Theorem 13.1 would be to consider instead of (11.2) the sum of independent random matrices $B_{n\times n} + n^{-1}\sum_{j=1,\dots,n}\Xi_{n\times n}^{(j)}$.

Let

$$\Xi^{(j)}_{n\times n} = \left[\xi^{(n,j)}_{pl}\right]^{l=1,\dots,n}_{p=1,\dots,n}, \mathbf{E}\,\Xi^{(j)}_{n\times n} = A^{(j)}_{n\times n}, j=1,...,n$$

be independent Hermitian random matrices. Let $B_{n\times n}, n=1,2,\dots,$ be Hermitian matrices.

Consider the Stieltjes transform

$$\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\mu_n\left(x, B_{n\times n} + n^{-1}\sum_{j=1,\dots,n}\Xi^{(j)}_n\right)}{x-z} = n^{-1}\mathrm{Tr}\left[B_{n\times n} + \frac{1}{n}\sum_{j=1,\dots,n}\Xi^{(j)}_n - zI_n\right]^{-1},$$

where $z=t+\mathrm{i}s,\ \ s>0$ and the canonical equation $K_{27}$ for Hermitian matrix $Q_n(z)=\{q_{pl}(z)\}$

$$Q_n(z) = \left\{B_n - zI_n + \frac{1}{n}\sum_{k=1,\dots,n} A^{(k)}_n - \frac{1}{n^2}\sum_{k=1,\dots,n}\mathbf{E}\,[\Xi^{(k)}_n - A^{(k)}_n]Q_n(z)[\Xi^{(k)}_n - A^{(k)}_n]\right\}^{-1}. \tag{19.1}$$

Moreover

$$q^{(n)}_{pl}(z) = \int\limits_{-\infty}^{\infty}(x-z)^{-1}\mathrm{d}F_{p,l,n}(x), p,l=1,...,n,$$

where $F_{p,l,n}(x)$ are certain functions of bounded variation and $F_{p,p,n}(x)$ ara certain distribution functions, and these equalities help us to immediately establish that this equation has a unique solution in the class of analytical entries of the matrix $Q_n(z), \Im z \neq 0$.

Let

$$Q_n = \left[B_{n\times n} + n^{-1}\sum_{j=1,\dots,n}\Xi^{(j)}_n - zI_n\right]^{-1}, Q^{(j)}_n = \left[B_{n\times n} + n^{-1}\sum_{i=1,\dots,n,i\neq j}\Xi^{(i)}_n - zI_n\right]^{-1},$$
$$P_n = \left\{B_{n\times n} + n^{-1}\sum_{j=1,\dots,n} A^{(j)}_n - zI_n - n^{-2}\mathbf{E}\sum_{j=1,\dots,n} Y^{(j)}_n\mathbf{E}\,Q_nY^{(j)}_n\right\}^{-1}, \tag{19.2}$$

where $Y^{(j)}_n = \Xi^{(j)}_n - A^{(j)}_n$.

We introduce the following conditions:

$$\max_{j,k=1,\dots,n} n^{-1}\lambda_j\{\mathbf{E}\,[Y^{(k)}_n]^2\} \leq c, \lim_{n\to\infty}\max_{k=1,\dots,n} n^{-1}\mathrm{Tr}\,[A^{(k)}_{n\times n}]^4 \leq c, \lim_{n\to\infty}\max_{k=1,\dots,n} n^{-3}\mathrm{Tr}\,\mathbf{E}\,[\Xi^{(k)}_n]^4 \leq c \tag{19.3}$$

for any are complex symmetrical matrices $\Theta^{(i)}_n, i=1,2$ with bounded singular eigenvalues

$$\lim_{n\to\infty}\sup_{\Theta^{(1)}_n:||\Theta^{(1)}_n||\leq 1}\max_{k=1,\dots,n} n^{-3}\mathbf{E}\,|\mathrm{Tr}\,Y^{(k)}_n\Theta^{(1)}_n|^2 = 0, \tag{19.4}$$

$$\lim_{n\to\infty}\frac{1}{n^3}\max_{k,j=1,\dots,n,j\neq k}\sup_{\Theta^{(i)}_n:||\Theta^{(i)}_n||\leq 1, i=1,2}\left|\mathbf{E}\,\mathrm{Tr}\,\Theta^{(1)}_nY^{(k)}_n\Theta^{(2)}_nY^{(j)}_n\Theta^{(2)}_nY^{(k)}_n\Theta^{(2)}_nY^{(j)}_n\right| = 0. \tag{19.5}$$

This condition may seem cumbersome, but there are several simple cases when it is satisfied. Let's look at one simple example.

**Remark 19.1.** *Let the non-coinciding entries of the matrices $Y^{(k)}_n$ be independent and their fourth moments be bounded. Then conditions (19.3)–(19.5) are valid.*

**Theorem 19.2.** *(Canonical equation $K_{27}$) Suppose that conditions (19.3)–(19.5) are satisfied. Then, for almost all $x$, with probability one,*

$$\lim_{n\to\infty}\left|\mu_n\left(x, B_{n\times n}+n^{-1}\sum_{j=1,\dots,n}\Xi_n^{(j)}\right)-\mu_n(x)\right|=0,$$

*where $\mu_n(x)$ is a distribution function whose Stieltjes transform satisfies the relation*

$$\int_{-\infty}^{\infty}(x-z)^{-1}\mathrm{d}\mu_n(x)=n^{-1}\mathrm{Tr}\,F_n(z),$$

*and the matrix $F_n(z)=\{f_{pl}(z)\}, p,l=1,...,n$ is the solution of the canonical equation (19.1). There exists a unique solution $F_n(z)$ of the canonical equation (19.1) in the class of analytic matrix functions*

$$L=\{F_n(z):\Im F_n(z)>0,\Im z>0\},$$

*and*

$$f_{pl}(z)=\int_{-\infty}^{\infty}(u-z)^{-1}\,\mathrm{d}G_{pl}(u),$$

*where $G_{pl}(u), p\neq l$ are functions of bounded variation and $G_{pp}(u)$ are distribution functions.*

*Proof.* We follow several steps of our proof:

# 20 Self-averaging of normalized spectral functions. The main statement of the REFORM method

**Lemma 20.1.** *(The main statement of the REFORM method)*[1] *If, for each $n$, the matrices $\Xi_{n\times n}^{(k)}, k=1,...,n$ are independent and defined in a common probability space, and if the conditions (19.3)–(19.5) are fulfilled then , for almost all $x$*

$$\mathrm{l.i.m.}_{n\to\infty}\left|\mu_n\left(x, B_{n\times n}+n^{-1}\sum_{j=1,\dots,n}\Xi_n^{(j)}\right)-\Phi_n(x)\right|=0,$$

*where $\Phi_n(x)$ is a distribution function whose Stieltjes transform satisfies the relation*

$$\int_{0}^{\infty}(x-z)^{-1}\mathrm{d}\Phi_n(x)=n^{-1}\mathbf{E}\,\mathrm{Tr}\left(B_{n\times n}+n^{-1}\sum_{j=1,\dots,n}\Xi_n^{(j)}-zI_n\right)^{-1}.$$

*Proof.* Denote $\gamma_k=\mathbf{E}\left[\mathrm{Tr}Q_n|\sigma_{k-1}\right]-\mathbf{E}\left[\mathrm{Tr}Q_n|\sigma_k\right],\ k=1,\dots,n$, where (see (19.2))

$$Q_n=\left(B_n+n^{-1}\sum_{k=1,\dots,n}\Xi_{n\times n}^{(k)}-zI_n\right)^{-1}$$

and $\sigma_k$ is the smallest $\sigma$-algebra generated by the matrices $\Xi_{n\times n}^{(s)},\ s=k+1,\dots,n$. This enables us to write

$$\mathrm{Tr}Q_n-\mathbf{E}\mathrm{Tr}Q_n=\sum_{k=1}^{n}\gamma_k.$$

As in the corresponding proofs in [1–4] we get

$$\mathbf{E}\,|n^{-1}\mathrm{Tr}\,Q_n - n^{-1}\mathbf{E}\,\mathrm{Tr}\,Q_n|^2 = n^{-2}\sum_{k=1}^{n}\mathbf{E}\,\gamma_k^2,$$

$$\begin{aligned}
\mathbf{E}\,|\gamma_k|^2 = \quad & \mathbf{E}\left|\mathbf{E}\big[\mathrm{Tr}\,Q - Q^{(k)}|\sigma_{k-1}\big] - \mathbf{E}\left[\mathrm{Tr}\,Q - Q^{(k)}|\sigma_k\right]\right|^2 \\
& = \mathbf{E}\left|\mathbf{E}\big[\mathrm{Tr}[-n^{-1}Q^{(k)}Y_n^{(k)}Q^{(k)} + n^{-2}Q^{(k)}\Xi_n^{(k)}Q^{(k)}\Xi_n^{(k)}Q]|\sigma_{k-1}\big]\right. \\
& \left. -\mathbf{E}\left[\mathrm{Tr}[-n^{-1}Q^{(k)}Y_n^{(k)}Q^{(k)} + n^{-2}Q^{(k)}\Xi_n^{(k)}Q^{(k)}\Xi_n^{(k)}Q]|\sigma_k\right]\right|^2 \\
& \le 2n^{-2}\mathbf{E}\,|\mathrm{Tr}\,Q^{(k)}Y_n^{(k)}Q^{(k)}|^2 + 2n^{-4}\mathbf{E}\,|\mathrm{Tr}\,Q^{(k)}\Xi_n^{(k)}Q^{(k)}\Xi_n^{(k)}Q|^2 \\
& \le cn^{-2}\mathbf{E}\,|\mathrm{Tr}\,Q^{(k)}Y_n^{(k)}Q^{(k)}|^2 + cn^{-4}\mathbf{E}\,\mathrm{Tr}\,Q^{(k)}[\Xi_n^{(k)}]^2Q^{(k)*}\mathbf{E}\,\mathrm{Tr}\,Q^{(k)}\Xi_n^{(k)}QQ^*\Xi_n^{(k)}Q^{(k)*} \\
& \le cn^{-2}\mathbf{E}\,|\mathrm{Tr}\,Q^{(k)}Y_n^{(k)}Q^{(k)}|^2 + cn^{-4}[\mathbf{E}\,\mathrm{Tr}\,[\Xi_n^{(k)}]^2]^2 \\
& \le cn^{-2}\sup_{\Theta_n:||\Theta_n||\le 1}\mathbf{E}\,|\mathrm{Tr}\,\Theta_n Y_n^{(k)}|^2 + cn^{-4}[\mathbf{E}\,\mathrm{Tr}\,[\Xi_n^{(k)}]^2]^2 \le c.
\end{aligned} \tag{20.1}$$

Using (19.4) we have

$$\begin{aligned}
& \lim_{n\to\infty}\mathbf{E}\,|n^{-1}\mathrm{Tr}\,Q_n - n^{-1}\mathbf{E}\,\mathrm{Tr}\,Q_n|^2 \le \lim_{n\to\infty} n^{-1}\max_{k=1,\dots,n}\mathbf{E}\,|\gamma_k|^2 \\
& \le \lim_{n\to\infty}\max_{k=1,\dots,n}\left\{\sup_{\Theta_n^{(1)}:||\Theta_n^{(1)}||\le 1} n^{-3}\mathbf{E}\,|\mathrm{Tr}\,[\Xi_n^{(k)} - A_n^{(k)}]\Theta_n^{(1)}|^2\right. \\
& \left. + cn^{-5}[\mathbf{E}\,\mathrm{Tr}\,[\Xi_n^{(k)}]^2]^2\right\} = 0.
\end{aligned} \tag{20.2}$$

Hence, by using the inverse Stieltjes trahsform transform and (20.1)–(20.2), we can complete the proof of Lemma 20.1. □

## 21 The main equality

In order to simplify the formulas, we will omit the symbol $(z)$. Let us prove the main statement.

**Lemma 21.1.** *Under the conditions of Theorem 19.2 for any* $z, \Im z \ne 0$ $\mathrm{l.i.m.}_{n\to\infty} n^{-1}|\mathrm{Tr}\,[P_n(z) - Q_n(z)]| = 0$.

*Proof.* Using notations (19.2) after some transforms we arrive at the following equation taking into account that $\Xi_n^{(k)}$ is stochastically independent of $Q_n^{(k)}$

$$\frac{1}{n}\mathrm{Tr}\,[P_n - \mathbf{E}\,Q_n] = \frac{1}{n}\mathrm{Tr}\,P_n\mathbf{E}\left(\frac{1}{n}\sum_{k=1}^{n}\Xi_n^{(k)} - \frac{1}{n}\sum_{k=1}^{n}A_n^{(k)} + \frac{1}{n^2}\sum_{k=1}^{n}\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}]\right)Q_n$$
$$= \frac{1}{n^2}\sum_{k=1}^{n}\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n + \frac{1}{n^3}\sum_{k=1}^{n}\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n$$
$$= \frac{1}{n^2}\sum_{k=1}^{n}\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}][Q_n - Q_n^{(k)}] + \frac{1}{n^3}\sum_{k=1}^{n}\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n$$
$$= -\frac{1}{n^3}\sum_{k=1}^{n}\{\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}]Q_n$$
$$+\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}A_n^{(k)}[Q_n - Q_n^{(k)}]\} + \frac{1}{n^3}\sum_{k=1}^{n}\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n$$
$$= -\frac{1}{n^3}\sum_{k=1}^{n}\{\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}][Q_n - Q_n^{(k)}] +$$
$$+\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)} + \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}A_n^{(k)}[Q_n - Q_n^{(k)}]\}$$
$$+\frac{1}{n^3}\sum_{k=1}^{n}\mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n. \tag{21.1}$$

Using equalities

$$\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}$$
$$= \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}][Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)}][\Xi_n^{(k)} - A_n^{(k)}][Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)}] + \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n^{(k)},$$
$$\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n = \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}][\mathbf{E}\,Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)}][\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n$$
$$+\mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n[\Xi_n^{(k)} - A_n^{(k)}][\mathbf{E}\,Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)}] + \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}][\mathbf{E}\,Q_n^{(k)}$$

we have from this equality (21.1)

$$\frac{1}{n}\mathrm{Tr}\,[P_n - \mathbf{E}\,Q_n] = L_1 + L_2 + L_3 + L_4 + L_5,$$

$$
\begin{aligned}
L_1 &= -\frac{2}{n^3}\sum_{k=1}^{n} \operatorname{Tr} P_n \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}A_n^{(k)}[Q_n - Q_n^{(k)}],\\
L_2 &= -\frac{2}{n^3}\sum_{k=1}^{n} \operatorname{Tr} P_n \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}][Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)}][\Xi_n^{(k)} - A_n^{(k)}][Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)}],\\
L_3 &= -\frac{1}{n^3}\sum_{k=1}^{n} \operatorname{Tr} P_n \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}][Q_n - Q_n^{(k)}],\\
L_4 &= -\frac{1}{n^3}\sum_{k=1}^{n} \operatorname{Tr} P_n \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n^{(k)}[\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,[Q_n^{(k)} - Q_n],\\
L_5 &= -\frac{1}{n^3}\sum_{k=1}^{n} \operatorname{Tr} P_n \mathbf{E}\,[\Xi_n^{(k)} - A_n^{(k)}][\mathbf{E}\,Q_n^{(k)} - \mathbf{E}\,Q_n][\Xi_n^{(k)} - A_n^{(k)}]\mathbf{E}\,Q_n,\\
[Q_n - \mathbf{E}\,Q_n] &= \sum_{j=1, j\neq k}^{n} \{\mathbf{E}\,(Q_n|\sigma_{j-1}, \Xi^{(k)}) - \mathbf{E}\,(Q_n|\sigma_j, \Xi^{(k)})\},\\
Q_n^{(k)} - \mathbf{E}\,Q_n^{(k)} &= \sum_{j=1, j\neq k}^{n} \{\mathbf{E}\,(Q_n^{(k)}|\sigma_{j-1}) - \mathbf{E}\,(Q_n^{(k)}|\sigma_j)\},
\end{aligned}
\tag{21.2}
$$

$\sigma_j$ is the smallest $\sigma$-algebra generated by the matrices $\Xi_{n\times n}^{(s)}$, $s = j+1, \ldots, n, k$.

Let's start the analysis of these quantities with the simplest one $L_1$. It's obvious that due to the conditions (19.3)–(19.5) that

$$
\begin{aligned}
|L_1| \quad &= \frac{1}{n^4}\left|\sum_{k=1}^{n} \mathbf{E}\operatorname{Tr} P_n[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}A_n^{(k)}Q_n^{(k)}\Xi_n^{(k)}Q_n]\right|\\
&\le \frac{1}{n^4}\sum_{k=1}^{n}\sqrt{\mathbf{E}\operatorname{Tr} P_n[\Xi_n^{(k)} - A_n^{(k)}]Q_n^{(k)}Q_n^{(k)*}[\Xi_n^{(k)} - A_n^{(k)}]P_n^*}\\
&\times\sqrt{\mathbf{E}\operatorname{Tr} A_n^{(k)}Q_n^{(k)}\Xi_n^{(k)}Q_nQ_n^*\Xi_n^{(k)}Q_n^{(k)*}A_n^{(k)}}\\
&\le \frac{c}{n^3}\max_{k=1,\ldots,n}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)} - A_n^{(k)}]^2}\{\operatorname{Tr}[A_n^{(k)}]^4\}^{1/4}\{\mathbf{E}\operatorname{Tr} Q_n^{(k)}[\Xi_n^{(k)}]^2Q_n^{(k)}Q_n^{(k)*}[\Xi_n^{(k)}]^2Q_n^{(k)*}\}^{1/4}\\
&\le \frac{c}{n^3}\max_{k=1,\ldots,n}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)} - A_n^{(k)}]^2}\{\operatorname{Tr}[A_n^{(k)}]^4\}^{1/4}\{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)}]^4\}^{1/4}\\
&\le \frac{c}{\sqrt{n}}.
\end{aligned}
\tag{21.3}
$$

Similarly we obtain

$$
\begin{aligned}
|L_3| &\le \frac{c}{n^3}\max_{k=1,\ldots,n}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)} - A_n^{(k)}]^4}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)}]^2} \le \frac{c}{\sqrt{n}},\\
|L_4| &\le \frac{c}{n^3}\max_{k=1,\ldots,n}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)} - A_n^{(k)}]^4}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)}]^2} \le \frac{c}{\sqrt{n}},\\
|L_5| &\le \frac{c}{n^3}\max_{k=1,\ldots,n}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)} - A_n^{(k)}]^4}\sqrt{\mathbf{E}\operatorname{Tr}[\Xi_n^{(k)}]^2} \le \frac{c}{\sqrt{n}}.
\end{aligned}
\tag{21.4}
$$

We now move on to the most important part of our proof and consider the equality

$$
\begin{aligned}
Q_n^{(k)}-\mathbf{E}\,Q_n^{(k)} \quad &= \sum_{j=1,\dots,n,j\neq k}[\mathbf{E}\,(Q_n^{(k)}-Q_n^{(k,j)}|\sigma_{j-1})-\mathbf{E}\,(Q_n^{(k)}-Q_n^{(k,j)}|\sigma_j)]=M_1^{(k)}+M_2^{(k)},\\
&M_1^{(k)}=\frac{1}{n}\sum_{j=1,\dots,n,j\neq k}[\mathbf{E}\,(Q_n^{(k,j)}[\Xi_n^{(j)}-A_n^{(j)}]Q_n^{(k,j)}|\sigma_{j-1})]\\
&M_2^{(k)}=\frac{1}{n^2}\sum_{j=1,\dots,n,j\neq k}[\mathbf{E}\,(Q_n^{(k,j)}\Xi_n^{(j)}Q_n^{(k,j)}\Xi_n^{(j)}Q_n^{(k)}|\sigma_{j-1})\\
&-\mathbf{E}\,(Q_n^{(k,j)}\Xi_n^{(j)}Q_n^{(k,j)}\Xi_n^{(j)}Q_n^{(k)}|\sigma_j)]. \qquad (21.5)
\end{aligned}
$$

Then we get

$$
\begin{aligned}
L_2= \quad & -\frac{2}{n^3}\sum_{k=1}^n \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)}-A_n^{(k)}][Q_n^{(k)}-\mathbf{E}\,Q_n^{(k)}][\Xi_n^{(k)}-A_n^{(k)}][Q_n^{(k)}-\mathbf{E}\,Q_n^{(k)}]\\
&=-\frac{2}{n^3}\sum_{k=1}^n \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)}-A_n^{(k)}][M_1^{(k)}+M_2^{(k,j)}][\Xi_n^{(k)}-A_n^{(k)}][M_1^{(k)}+M_2^{(k)}]\\
&=T_1+T_2+T_3, \qquad (21.6)
\end{aligned}
$$

where

$$
\begin{aligned}
&T_1=-\frac{2}{n^3}\sum_{k=1}^n \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)}-A_n^{(k)}][M_1^{(k,j)}][\Xi_n^{(k)}-A_n^{(k)}][M_1^{(k)}],\\
&T_2=-\frac{2}{n^3}\sum_{k=1}^n \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)}-A_n^{(k)}][M_1^{(k)}][\Xi_n^{(k)}-A_n^{(k)}][M_2^{(k)}]\\
&-\frac{2}{n^3}\sum_{k=1}^n \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)}-A_n^{(k)}]M_2^{(k)}[\Xi_n^{(k)}-A_n^{(k)}][M_1^{(k)}],\\
&T_3=-\frac{2}{n^3}\sum_{k=1}^n \mathrm{Tr}\,P_n\mathbf{E}\,[\Xi_n^{(k)}-A_n^{(k)}]M_2^{(k)}[\Xi_n^{(k)}-A_n^{(k)}]M_2^{(k)}. \qquad (21.7)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|T_1| \quad &\le \frac{c}{n^3}\max_{k,j=1,\dots,n,j\neq k}\Big|\mathbf{E}\,\mathrm{Tr}\,P_n[\Xi_n^{(k)}-A_n^{(k)}][\mathbf{E}\,(Q_n^{(k,j)}[\Xi_n^{(j)}-A_n^{(j)}]Q_n^{(k,j)}|\sigma_{j-1})]\\
&\times[\Xi_n^{(k)}-A_n^{(k)}][\mathbf{E}\,(Q_n^{(k,j)}[\Xi_n^{(j)}-A_n^{(j)}]Q_n^{(k,j)}|\sigma_{j-1})]\Big|\\
&\le \frac{2}{n^3}\max_{k,j=1,\dots,n,j\neq k}\ \max_{\Theta_n^{(i)}:||\Theta_n^{(i)}||\le 1,i=1,\dots,3}|\mathbf{E}\,\mathrm{Tr}\,\Theta_n^{(1)}[\Xi_n^{(k)}-A_n^{(k)}]\Theta_n^{(2)}[\Xi_n^{(j)}-A_n^{(j)}]\\
&\times\Theta_n^{(2)}[\Xi_n^{(k)}-A_n^{(k)}]\Theta_n^{(2)}[\Xi_n^{(j)}-A_n^{(j)}]|,
\end{aligned}
$$

where $\Theta_n^{(i)}$ are complex symmetrical matrices with bounded singular eigenvalues.

Similarly we get using inequality $\mathrm{Tr}\,AB\le\max_k\lambda_k\{A\}\mathrm{Tr}\,B$ where $A,B$ are positive definite Hermitian matrices and conditions (19.3) and (19.5)

$$\begin{aligned}
|T_2| \leq \quad & \Bigg| \frac{2}{n^4} \max_{k,j=1,\dots,n, j\neq k} |\mathbf{E}\,\mathrm{Tr}\, P_n[\Xi_n^{(k)} - A_n^{(k)}][\mathbf{E}\,(Q_n^{(k,j)}[\Xi_n^{(j)} - A_n^{(j)}]Q_n^{(k,j)}|\sigma_{j-1})] \\
& \times [\Xi_n^{(k)} - A_n^{(k)}](\mathbf{E}_{j-1} - \mathbf{E}_j)Q_n^{(k,j)}\Xi_n^{(j)}Q_n^{(k,j)}\Xi_n^{(j)}Q_n^{(k)} \Bigg| \\
& \leq \frac{c}{n^4} \max_{k,j=1,\dots,n, j\neq k} \sqrt{\mathbf{E}\,\mathrm{Tr}\, Q_n^{(k)} P_n Y_n^{(k)} Q_n^{(k,j)} Y_n^{(j)} Q_n^{(k,j)} Q_n^{(k,j)*} Y_n^{(j)} Q_n^{(k,j)*} Y_n^{(k)} (Q_n^{(k)} P_n)^*} \\
& \times \sqrt{\mathbf{E}\,\mathrm{Tr}\,[Y_n^{(k)}]^2 Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} [\Xi_n^{(j)}]^2 Q_n^{(k,j)*} \Xi_n^{(j)} Q_n^{(k,j)*}} \\
& \leq \frac{c}{n^4} \max_{k,j=1,\dots,n, j\neq k} \{\mathbf{E}\,\mathrm{Tr}\,[Y_n^{(k)}]^4\}^{1/4} \{\mathbf{E}\,\mathrm{Tr}\,[Y_n^{(j)}]^4\}^{1/4} \sqrt{\max_k \lambda_k \{\mathbf{E}\,[Y_n^{(k)}]^2\}} \\
& \times \sqrt{\mathbf{E}\,\mathrm{Tr}\, Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} [\Xi_n^{(j)}]^2 Q_n^{(k,j)*} \Xi_n^{(j)} Q_n^{(k,j)*}} \\
& \leq \frac{c}{n^4} \max_{k,j=1,\dots,n, j\neq k} \{\mathbf{E}\,\mathrm{Tr}\,[Y_n^{(k)}]^4\}^{1/4} \{\mathbf{E}\,\mathrm{Tr}\,[Y_n^{(j)}]^4\}^{1/4} \sqrt{\max_k \lambda_k \{\mathbf{E}\,[Y_n^{(k)}]^2\}} \{\mathbf{E}\,\mathrm{Tr}\,[\Xi_n^{(k)}]^4\}^{1/2} \\
& \leq \frac{c}{n^{1/2}},
\end{aligned}$$

where $(\mathbf{E}\,|\sigma_j) = \mathbf{E}_j$,

$$\begin{aligned}
|T_3| \leq \quad & \Bigg| \frac{2}{n^5} \max_{k,j=1,\dots,n, j\neq k} |\mathbf{E}\,\mathrm{Tr}\, P_n Y_n^{(k)} (\mathbf{E}_{j-1} - \mathbf{E}_j) Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k)} \\
& \times Y_n^{(k)} (\mathbf{E}_{j-1} - \mathbf{E}_j) Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k)} \Bigg| \\
& \leq \frac{c}{n^5} \max_{k,j=1,\dots,n, j\neq k} \Bigg\{ \mathbf{E}\,\mathrm{Tr}\,[P_n Y_n^{(k)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k)}] \\
& \times [P_n Y_n^{(k)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k)}]^* \Bigg\}^{1/2} \\
& \times \sqrt{\mathbf{E}\,\mathrm{Tr}\,[Y_n^{(k)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k)}][Y_n^{(k)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k,j)} \Xi_n^{(j)} Q_n^{(k)}]^*} \\
& \leq \frac{c}{n^5} \max_{k,j=1,\dots,n, j\neq k} \max_k \lambda_k \{\mathbf{E}\,[Y_n^{(k)}]^2\} \mathbf{E}\,\mathrm{Tr}\,[\Xi_n^{(k)}]^4 \\
& \leq \frac{c}{n}.
\end{aligned}$$

We have from these equalities (21.1)–(21.7)

$$\frac{1}{n}\mathbf{E}\,|\mathrm{Tr}\,[P_n - \mathbf{E}\,Q_n]|^2 \leq \frac{c}{\sqrt{n}}.$$

Therefore we complete the proof of Lemma 21.1. □

As in the proof of Theorem 13.1 repeating the proof of Lemma 21.1 we have for any matrix $\Theta_n$ with bounded singular eigenvalues

$$\begin{aligned}
\frac{1}{n}\mathrm{Tr}\,\Theta_n \mathbf{E}\,Q_n(z) \quad & = \frac{1}{n}\mathrm{Tr}\,\Theta_n \Bigg\{ -zI_n + B_n + \frac{1}{n} \sum_{k=1,\dots,n} A_n^{(k)} \\
& - \frac{1}{n^2} \sum_{k=1,\dots,n} \mathbf{E}\,Y_n^{(k)} \mathbf{E}\,Q_n(z) Y_n^{(k)} \Bigg\}^{-1} + \frac{1}{n}\mathrm{Tr}\,\Theta_n E_n, \qquad (21.8)
\end{aligned}$$

where $\lim_{n\to\infty} |\frac{1}{n}\mathrm{Tr}\,\Theta_n E_n| = 0$.

# 22 The $G$-Matrix Expansion Method

However, we still have an equation for a matrix $\mathbf{E}\, Q_{2n}$ with some error matrix $E_n$ wose entries tend to zero when $n$ tends to infinity. For the first time, it was proved in [4] that it is possible to find an equation without this error that will approximate the matrix $\mathbf{E}\, Q_{2n}$ well. We called the procedure for finding this equation *the Matrix Expansion Method*, the idea of which is quite simple. We consider block matrices

$$W_{ns} = \left\{ \frac{1}{\sqrt{s}} n^{-1} \sum_{p=1,\dots,n} \Xi_{ns}^{(ij),p} \right\}_{i,j=1,\dots,s}, V_{ns} = \left\{ \delta_{i,j} [n^{-1} \sum_{p=1,\dots,n} A_n^{(k)} + B_n] \right\}_{i,j=1,\dots,s}$$

and

$$K_{ns}(z) = \{K_n^{(ij)}\}_{i,j=1,\dots,s} = [-I_{ns} z + V_{ns} + W_{ns}]^{-1},$$

where the matrices $\Xi_{2n}^{(ij),p}, i, j = 1, ..., m$ are independent and distributed in the same way as the matrix $\Xi_n^{(p)}$. Now the parameter $n$ is fixed and the limit is considered already when $s$ tends to infinity. Repeating the proof of Lemma 17.1, we get that there exists a certain such subsequence of parameter $s'$ that

$$\lim_{s' \to \infty} \frac{1}{s'} \sum_{i=1,\dots,s'} \mathbf{E}\, K_n^{(ii)}(z) = G_n(z)$$

and matrix $G_n(z) = \{g_{pl}(z)\}$ satisfies the canonical equation

$$G_n(z) = \left\{ B_{n \times n} - z I_n + n^{-1} \sum_{k=1,\dots,n} A_n^{(k)} - n^{-2} \sum_{k=1,\dots,n} \mathbf{E}\, [\Xi_n^{(k)} - A_n^{(k)}] G_n(z) [\Xi_n^{(k)} - A_n^{(k)}] \right\}^{-1}. \quad (22.1)$$

Moreover

$$g_{pl} n(z) = \int_{-\infty}^{\infty} (x - z) \mathrm{d} F_{p,l,n}(x), p, l = 1, ..., n,$$

where $F_{p,l,n}(x)$ are certain functions of bounded variation and $F_{p,p,n}(x)$ are certain distribution functions, which makes it possible to immediately establish that this equation (22.1) has a unique solution in the class of analytical entries of the matrix $G_n(z), \Im z \neq 0$.

# 23 The solution of the canonical equation $K_{27}$ is unique in the class of analytic matrix-functions

Let us prove that the solution of the canonical equation (22.1) is unique in the class of analytic matrix-functions $L$. Assume that there exist two solutions $C^{(1)}(z)$ and $C^{(2)}(z)$ from the class $L$ that they do not coincide at least at one point $z$

$$C^{(1)}(z) - C^{(2)}(z) = C^{(1)}(z) \mathbf{E}\, n^{-2} \sum_{k=1,\dots,n} \mathbf{E}\, Y_n^{(k)} [C^{(1)}(z) - C^{(2)}] Y_n^{(k)} C^{(2)}(z), Y_n^{(k)} = \Xi_n^{(k)} - A_n^{(k)}. \quad (23.1)$$

Let $C^{(1)}(z) - C^{(2)}(z) = U(z) \Lambda(z) V(z)$, where $U(z), V_n(z)$ are the Unitary matrices and $\Lambda_n(z) = \{\lambda_i(z) \delta_{ij}\}$ is the diagonal matrix of its singular eigenvalues. Then we have from this equation (23.1) using equality

$$|\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,A_n^{(k)}B_n^{(k)}|\le[\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,A_n^{(k)}A_n^{(k)*}]^{1/2}[\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,B_n^{(k)}B_n^{(k)*}]^{1/2}$$

and condition (19.3)

$$\begin{aligned} n^{-1}\sum_{j=1,\dots,n}\lambda_j(z) &= n^{-1}\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,U(z)^*C^{(1)}(z)n^{-2}Y_n^{(k)}U(z)\Lambda(z)V(z)Y_n^{(k)}C^{(2)}(z)V^*(z)\\ &\le n^{-1}\{\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,[U(z)^*C^{(1)}(z)n^{-2}Y_n^{(k)}U(z)\Lambda^{1/2}(z)]\\ &\times[U(z)^*C^{(1)}(z)n^{-2}Y_n^{(k)}U(z)\Lambda^{1/2}(z)]^*\}^{1/2}\\ &\times\{\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,[\Lambda(z)^{1/2}V(z)Y_n^{(k)}C^{(2)}(z)V^*(z)][\Lambda(z)^{1/2}V(z)Y_n^{(k)}C^{(2)}(z)V^*(z)]^*\}^{1/2}\\ &\le c|\Im z|^{-2}\max_{j=1,\dots,n}n^{-1}\lambda_j\{\mathbf{E}\,[Y_n^{(k)}]^2\}n^{-1}\sum_{j=1,\dots,n}\lambda_j(z)\le c|\Im z|^{-2}n^{-1}\sum_{j=1,\dots,n}\lambda_j(z). \end{aligned}$$

Therefore, these two solutions coincide for $c\,(\mathrm{Im}\,z)^{-2}<1$. Since the entries of the matrices $C^{(1)}(z)$ and $C^{(2)}(z)$ are analytic functions from $L$, these solutions coincide for all $z:\ \mathrm{Im}z>0$. Thus, the uniqueness of the solution of the canonical equation $K_{27}$ is proved for all $z:\mathrm{Im}\,z>0$ and this unique solution can be represented as

$$q_{pl}^n(z)=\int_{-\infty}^{\infty}(x-z)\mathrm{d}F_{p,l,n}(x),p,l=1,...,n.$$

□

As in the section 18 we continue the proof of theorem 19.2.

# 24 Approximation by canonical equation $K_{27}$

So, we have obtained two systems of canonical equations (21.8) and (22.1) and for their solutions we have

$$\mathbf{E}\,Q_n(z)-G_n(z)=\mathbf{E}\,Q_n(z)\mathbf{E}\,n^{-2}\sum_{k=1,\dots,n}\mathbf{E}\,Y_n^{(k)}[Q_n(z)-G_n(z)]Y_n^{(k)}G_n(z)+E_n,Y_n^{(k)}=\Xi_n^{(k)}-A_n^{(k)}.$$

Let $\mathbf{E}\,Q_n(z)-G_n(z)=U(z)\Lambda(z)V(z)$, where $U(z),V_n(z)$ are the Unitary matrices and $\Lambda_n(z)=\{\lambda_i(z)\delta_{ij}\}$ is the diagonal matrix of its singular eigenvalues. Then we have as in Section 23

$$\begin{aligned} n^{-1}\sum_{j=1,\dots,n}\lambda_j(z) &= n^{-1}\mathrm{Tr}\sum_{k=1,\dots,n}\mathbf{E}\,U(z)^*\mathbf{E}\,Q_n(z)n^{-2}Y_n^{(k)}U(z)\Lambda(z)V(z)Y_n^{(k)}G_n(z)V^*(z)\\ &+n^{-1}\mathrm{Tr}\,U(z)^*E_nV^*(z)\\ &\le c|\Im z|^{-2}\max_{j=1,\dots,n}n^{-1}\lambda_j\{\mathbf{E}\,[Y_n^{(k)}]^2\}n^{-1}\sum_{j=1,\dots,n}\lambda_j(z)\\ &\le c|\Im z|^{-2}n^{-1}\sum_{j=1,\dots,n}\lambda_j(z)+n^{-1}|\mathrm{Tr}\,U(z)^*E_nV^*(z)|. \end{aligned}$$

Therefore, under condition $c\,(\mathrm{Im}\,z)^{-2}<1$

$$\lim_{n\to\infty} n^{-1} \sum_{j=1,\dots,n} \lambda_j(z) = 0. \tag{24.1}$$

The functions $n^{-1}\mathbf{E}\,\mathrm{Tr}\, Q_n(z), n^{-1}\mathrm{Tr}\, G_n(z)$ are the Stieltjes transforms of some functions $F_n(x)$ of bounded variation and they are analytical functions in $z, \Im z > 0$. Therefore, for any convergent subsequence $F_{n'}(x)$ of these functions the difference $n^{-1}\mathbf{E}\,\mathrm{Tr}\, Q_n(z) - n^{-1}\mathrm{Tr}\, G_n(z)$ will converge to some analytical function. Then reasoning from the contrary choosing convergent subsequences due to the fact that the Stiltjes limit transforms are analytical functions in $z, \Im z > 0$ the limit (24.1) will be valid for any $z, \Im z > 0$. *Thus, we have proved the main statement of Theorem 19.2.*

# 25 The MAGIC estimator $G_{55}$ for a covariance matrix based on the canonical equation $K_{16}$

This is the main goal of our research. That is, we turn to the age-old problem of estimating a covariance matrix $R_n$ by independent observations $\vec{x}_k, k = 1, ..., n$ of corresponding vector $\vec{\xi}, \mathbf{E}\,\vec{\xi} = \vec{a}$. Let us immediately note that the most difficult case is when the vector $\vec{a}$ contains many components. Let us further recall that many problems are related to the analysis of such matrices, for example, in numerical analysis, multidimensional statistical analysis, etc. Moreover, in accordance with MAGIC, as a rule, these problems come down to finding some functions of these matrices $R_n$, for example, traces of their resolvents $\mathrm{Tr}\,[R_n + \epsilon I_n]^{-1}, \epsilon > 0$. Note that the resulting estimator $G_{55}$ has a complex form, but it can significantly reduce the number of necessary observations on the vector $\vec{\xi}$.

We move on to finding estimators of covariance matrices using canonical equations. We will show how this can be done using an example of the equation $K_{16}$. First we will find the relationship between the $G$-transform and the Fourier and the Laplace transforms.

# 26 $G$-transform

Let us find the inverse formula for the transform $\mathbf{E}\,(\alpha + \mathrm{i}x + \mathrm{i}\xi)^{-1}, \alpha \geq c > 0$ of the distribution function $F(x)$ of a random variable $\xi$, or the inverse formula for the transform $\mathbf{E}\,(\alpha + \mathrm{i}x + \xi)^{-1}, \alpha \geq c > 0$ of the distribution function of random variable $\xi > 0$. This transform is neither the Stieltjes transform (although close to it), nor the Gilbert transform (also close to him), nor the characteristic function (but you can reduce to it) and it is very important for our problems to estimate some functions of a covariance matrix. To avoid confusion, we call this transform as $G$-transform.

**Remark 26.1.** *We cannot analytically continue $G$-transform $\mathbf{E}\,(\alpha + \mathrm{i}x + \mathrm{i}\xi)^{-1}, \alpha \geq c > 0$ to the complex domain because our statements in this article are of the following form*

$$p \lim_{n\to\infty} \left\{ \int_{-\infty}^{\infty} \frac{1}{\alpha + \mathrm{i}x + \mathrm{i}u} \mathrm{d}F_n(u) - \Psi_n(\alpha + \mathrm{i}x) \right\} = 0, \alpha \geq c > 0$$

*and the functions $\Psi_n(\alpha+\mathrm{i}x)$ cannot be continued by a parameter $\alpha+\mathrm{i}x$ to the complex domain $z, \Im z > 0$. Therefore, we will take another way and give a new inverse formula for the $G$-transform.*

Using the Laplace transform we find its connection with $G$-transform

$$\mathbf{E}\,(\alpha + \mathrm{i}x + \xi)^{-1} = \int_0^\infty e^{-\mathrm{i}xs}\mathbf{E}\, e^{-s\xi - s\alpha}\mathrm{d}s, s > 0, \alpha \geq c > 0, \xi \geq 0. \tag{26.1}$$

Then using inverse Fourier transform we have

$$e^{s\alpha}(2\pi)^{-1}\int\limits_{-\infty}^{\infty} e^{\mathrm{i}xs}\mathbf{E}\,(\alpha+\mathrm{i}x+\xi)^{-1}\mathrm{d}x = \mathbf{E}\,e^{-s\xi}, s>0, \alpha\geq c>0, \xi\geq 0. \tag{26.2}$$

Sometimes it is more convenient to use the Fourier transform for $G$-transform $\mathbf{E}\,(\alpha+\mathrm{i}x+\mathrm{i}\xi)^{-1}, \alpha\geq c>0$ and $\alpha$ is a certain fixed parameter:

$$\Phi_1(x,\alpha) := \int\limits_{0}^{\infty} e^{\mathrm{i}xs}\mathbf{E}\,e^{\mathrm{i}s\xi}e^{-s\alpha}\mathrm{d}s = \mathbf{E}\,\frac{1}{-\mathrm{i}x-\mathrm{i}\xi+\alpha}, \alpha\geq c>0, \tag{26.3}$$

$$\Phi_2(x,\alpha) := \int\limits_{0}^{\infty} e^{\mathrm{i}xs}\mathbf{E}\,e^{-\mathrm{i}s\xi}e^{-s\alpha}\mathrm{d}s = \mathbf{E}\,\frac{1}{-\mathrm{i}x+\mathrm{i}\xi+\alpha}, \alpha\geq c>0. \tag{26.4}$$

**Remark 26.2.** *We draw the reader's attention to the presence of an arbitrary constant $\alpha>0$ in our transforms (26.1)–(26.4). This constant $\alpha$ plays a key role in our theory, but we then get rid of it with the help of our inverse transform. By the way, we revise the theory of regularization of complex systems with a small parameter $\epsilon$ and use an arbitrary regularised parameter $\alpha>0$( sometimes it is very big) which we then remove in the final formula of the successful solution of a problem.*

We can write the inverse formula for the Fourier transforms (26.3),(26.4) :

$$\mathbf{E}\,e^{\mathrm{i}s\xi} = \frac{1}{2\pi}\left\{\int\limits_{-\infty}^{\infty} e^{-\mathrm{i}xs}\Phi_1(x,\alpha)\mathrm{d}x\right\}\mathrm{d}xe^{s\alpha}, \mathbf{E}\,e^{-\mathrm{i}s\xi} = \frac{1}{2\pi}\left\{\int\limits_{-\infty}^{\infty} e^{-\mathrm{i}xs}\Phi_2(x,\alpha)\mathrm{d}x\right\}\mathrm{d}xe^{s\alpha}, s>0. \tag{26.5}$$

Using these formulas we can find the inversion formula for $G$-transform of the distribution function $F(u)=\mathbf{P}\,\{\xi<u\}, F(-\infty)=0$ at its points $u,v$ of continuity:

$$F(u)-F(v) = \frac{1}{2\pi}\lim_{c\to\infty}\left\{\int\limits_{c}^{c}\frac{e^{-\mathrm{i}vs}-e^{-\mathrm{i}us}}{\mathrm{i}s}K(s)\mathrm{d}s + \lim_{c\to\infty}\int\limits_{c}^{c}\frac{e^{-ivs}-e^{\mathrm{i}us}}{\mathrm{i}s}K(-s)\mathrm{d}s\right\}, \tag{26.6}$$

where $K(s)=\mathbf{E}\,e^{\mathrm{i}s\xi}$. This is exactly the formula we will use for statistical estimator $G_{57}$.

# 27 Convolution type integral equations

However for the MAGIC estimator $G_{55}$ we also obtain a Laplace transform $\int_0^\infty \exp\{-sy\}\mathrm{d}F(y)$ in our expressions with a real nonnegative parameter $s$ and we will not be able to analytically continue this transform to the complex plane for the reasons explained above. Note that we can always write this integral as

$$\int\limits_{0}^{\infty}\exp\{-|s|y\}\mathrm{d}F(y) = \mathbf{E}\int\limits_{0}^{\infty}\exp\{\mathrm{i}\eta sy\}\mathrm{d}F(y),$$

where $\eta$ is the random variable which is defined by Cauchy's law. Therefore, the inverse Fourier transform

$$p(x) := (2\pi)^{-1}\int\limits_{0}^{\infty} e^{-\mathrm{i}sx}\{\mathbf{E}\int\limits_{0}^{\infty}\exp\{\mathrm{i}\eta sy\}\mathrm{d}F(y)\}\mathrm{d}s$$

gives us the probability density $p(x)$ of the product of two random variables $\xi\eta$, where $\xi$ has distribution function $F(y)$ and these variables are independent. Then we obtain the convolution type integral equation for the sum of random variables $\ln\xi + \ln|\eta|$which, using the Fourier transform, can be written as

$$\int_{-\infty}^{\infty} e^{\mathrm{i}t\ln|x|}p(x)\mathrm{d}x = \mathbf{E}\, e^{\mathrm{i}t\ln\xi}\mathbf{E}\, e^{\mathrm{i}t\ln|\eta|}.$$

Hence

$$\mathbf{E}\, e^{\mathrm{i}t\ln\xi} = \frac{\int_{-\infty}^{\infty} e^{\mathrm{i}t\ln|x|}\left\{(2\pi)^{-1}\int_0^{\infty} e^{-\mathrm{i}sx}\{\mathbf{E}\int_0^{\infty}\exp\{\mathrm{i}\eta sy\}\mathrm{d}G(y)\}\mathrm{d}s\right\}\mathrm{d}x}{\mathbf{E}\, e^{\mathrm{i}t\ln|\eta|}}.$$

These transforms we have given for future studies, but in this article we will consider a class of random matrix functions in which the inversion formula for the Laplace transform is not used.

# 28 Definition of the estimator $G_{55}$. Stochastic canonical equation $K_{100}$. New regularization theory of a complex systems

Let the independent observations $\vec{x}_1, ..., \vec{x}_n$ of the $m_n$-dimensional random vector $\vec{\xi}$ be given,

$$\hat{R}_{m_n} := n^{-1}\sum_{k=1}^{n}(\vec{x}_k - \hat{\vec{x}})(\vec{x}_k - \hat{\vec{x}})^{\mathrm{T}}, \quad \hat{\vec{x}} = n^{-1}\sum_{k=1}^{n}\vec{x}_k.$$

The expression $\hat{R}_{m_n}$ is called an empirical covariance matrix. Typically, such a matrix $\hat{R}_{m_n}$ is used as a statistical estimator of the covariance matrix $R_{m_n} = \mathbf{E}\,(\vec{x}_k - \mathbf{E}\,\vec{x}_k)(\vec{x}_k - \mathbf{E}\,\vec{x}_k)^{\mathrm{T}}$. Many studies have been directed to finding estimators of this matrix $R_{m_n}$, but in many problems we do not need this matrix but some function of it, for example $\mathrm{Tr}\,[tI_n + R_{m_n}]^{-1}, t > 0$. It is often necessary to find an estimator of this expression. This is one of the principles of the MAGIC theory.

We use another estimator in MAGIC, which makes it possible to solve many problems when the main condition of this analysis is satisfied:

$$\lim_{n\to\infty} m_n n^{-1} < \infty.$$

This estimator is equal to

$$G_{55}(\alpha + \mathrm{i}x) = \left\{I_{m_n}(\alpha + \mathrm{i}x) + n^{-1}\sum_{k=1}^{n}(\vec{x}_k - \hat{\vec{x}}_{(k)})(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\tilde{\theta}_k^{-1}\right\}^{-1}, \tag{28.1}$$

where $\alpha > 0$ is a certain constant, $x$ is an arbitrary parameter, the random complex variables $\tilde{\theta}_k, \Re\tilde{\theta}_k > 0, k = 1, ..., n$ satisfy the system of stochastic canonical equations $K_{100}$

$$\tilde{\theta}_k + \frac{1}{n}(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\left\{I_{m_n}(\alpha + \mathrm{i}x) + \frac{1}{n}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})(\vec{x}_j - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\tilde{\theta}_j^{-1}\right\}^{-1}(\vec{x}_k - \hat{\vec{x}}_{(k)}) = 1, k = 1, .., n, \tag{28.2}$$

where $\hat{\vec{x}}_{(k)} = (n-1)^{-1}\sum_{j=1, j\neq k}^{n}\vec{x}_j$.

With the help of this estimator, many functions of the covariance matrix $R_{m_n}$ can be estimate. For clarity, we present an estimator of the Laplace transform of the spectral function of the covariance matrix $R_{m_n}$.

# 29 The properties of the solution of stochastic canonical equation $K_{100}$. Accompanying canonical equation $K_{101}$

**Lemma 29.1.** *If*

$$\max_{k=1,...,n} n^{-1}\vec{x}_k^{\mathrm{T}}\vec{x}_k \leq \rho_{\mathrm{magic}},$$

*and*

$$\alpha = \rho_{\mathrm{magic}} + \sqrt{\rho_{\mathrm{magic}}c^{-1}}, c < \min\{1, \rho_{\mathrm{magic}}^{-1}\}$$

*then the solutions $\tilde{\theta}_k, \Re\tilde{\theta}_k \geq 0, k = 1,..,n$ of the canonical equation $K_{100}$ satisfy inequalities*

$$\Re\tilde{\theta}_k \geq 1 - \rho_{\mathrm{magic}}\alpha^{-1} \geq c > 0, |\Im\tilde{\theta}_k| \leq \rho_{\mathrm{magic}}\alpha^{-1}, k = 1, ..., n, \tag{29.1}$$

*where $c$ is a positive constant.*

*Proof.* Denote $y = \Re\tilde{\theta}_k \geq 0, u = \Im\tilde{\theta}_k$. Then

$$y = 1 - n^{-1}(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}}W_n^{-1}(\vec{x}_k - \hat{\vec{x}}_{(k)}),$$

where

$$\begin{aligned}
W_n &= I_{m_n}\alpha + \frac{y}{y^2+u^2}n^{-1}(\vec{x}_1 - \hat{\vec{x}}_{(1)})(\vec{x}_1 - \hat{\vec{x}}_{(1)})^{\mathrm{T}} + C_{m_n} + D_{m_n},\\
C_{m_n} &= n^{-1}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})(\vec{x}_j - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\Re\tilde{\theta}_j^{-1},\\
D_{m_n} &= \left\{I_{m_n}x + n^{-1}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})(\vec{x}_j - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\Im\tilde{\theta}_j^{-1}\right\}\\
&\times\left(I_{m_n}\alpha + \frac{y}{y^2+u^2}n^{-1}(\vec{x}_1 - \hat{\vec{x}}_{(1)})(\vec{x}_1 - \hat{\vec{x}}_{(1)})^{\mathrm{T}} + C_{m_n}\right)^{-1}\\
&\times\left\{I_{m_n}x + n^{-1}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})(\vec{x}_j - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\Im\tilde{\theta}_j^{-1}\right\}.
\end{aligned}$$

We have from this equality $y \geq 1 - \rho_{\mathrm{magic}}\alpha^{-1} \geq c > 0$. Similarly we obtain $|\Im\tilde{\theta}_k| \leq \rho_{\mathrm{magic}}\alpha^{-1}, k = 1, ..., n$.

□

**Lemma 29.2.** *Under the conditions of Lemma 29.1 there exists complex a solution $\tilde{\theta}_k, k = 1, ..., n$ of the canonical equation $K_{100}$ with the non negative parts $\Re\tilde{\theta}_k \geq 1 - \rho_{\mathrm{magic}}\alpha^{-1} \geq c > 0, k = 1, ..., n,$ where $c$ is a positive constant.*

*Proof.* Let's notice at once that always at least one solution $\theta_k$ exists because there is no such value in the second part of the equation (28.2). Let's argue backwards. Suppose that at least one solution, say $\Re\tilde{\theta}_k \geq 0$, of the system (28.2) does not exist. The other possible solutions $\tilde{\theta}_j, \Re\tilde{\theta}_j \geq 0, j \neq k$ will be continues functions of this element $\Re\tilde{\theta}_k$ since for any $\epsilon$ and $j \neq k$

$$\max_{j\neq k}|\tilde\theta_j(\Re\tilde\theta_k,...) -\tilde\theta_j(\Re\tilde\theta_k-\epsilon,...)| \le \max_{j\neq k}\Bigg|\frac{1}{n}(\vec{x}_j-\hat{\vec{x}}_{(j)})^{\mathrm T}R_n(\tilde\theta_k) \times\left\{\frac{1}{n}\sum_{p=1,p\neq j,k}^{n}(\vec{x}_p-\hat{\vec{x}}_{(p)})(\vec{x}_p-\hat{\vec{x}}_{(p)})^{\mathrm T}\frac{\tilde\theta_j-\tilde\theta_j(\epsilon)}{\tilde\theta_j\tilde\theta_j(\epsilon)}\right\}R_n(\tilde\theta_k-\epsilon)(\vec{x}_j-\hat{\vec{x}}_{(j)}), +\frac{1}{n}(\vec{x}_j-\hat{\vec{x}}_{(j)})^{\mathrm T}R_n(\tilde\theta_k)(\vec{x}_k-\hat{\vec{x}}_{(k)})\frac{\epsilon}{\tilde\theta_k(\tilde\theta_k-\epsilon)}(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm T}R_n(\tilde\theta_k-\epsilon)(\vec{x}_j-\hat{\vec{x}}_{(j)})\Bigg|,$$

where $\tilde\theta_j(\epsilon)=\tilde\theta_j(\Re\tilde\theta_k-\epsilon,...)$,

$$R_n(\tilde\theta_k)=\left\{I_{m_n}(\alpha+\mathrm{i}x)+\frac{1}{n}\sum_{p=1,p\neq j,k}^{n}(\vec{x}_p-\hat{\vec{x}}_{(p)})(\vec{x}_p-\hat{\vec{x}}_{(p)})^{\mathrm T}\tilde\theta_p^{-1}+\frac{1}{n}(\vec{x}_k-\hat{\vec{x}}_{(k)})(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm T}\tilde\theta_k^{-1}\right\}^{-1}.$$

Again using inequalities $\Re\tilde\theta_k>1-\rho_{\mathrm{magic}}\alpha^{-1}, k=1,...,n$ we continue

$$\begin{aligned}&\max_{j\neq k}|\tilde\theta_j(\Re\tilde\theta_k,...)-\tilde\theta_j(\Re\tilde\theta_k-\epsilon,...)|\\ &\le\max_{j\neq k}\left\{(\vec{x}_j-\hat{\vec{x}}_{(j)})^{\mathrm T}R_n(\tilde\theta_k)\sum_{p=1,p\neq j}^{n}(\vec{x}_p-\hat{\vec{x}}_{(p)})(\vec{x}_p-\hat{\vec{x}}_{(p)})^{\mathrm T}|\tilde\theta_p|^{-2}R_n(\tilde\theta_k)^*(\vec{x}_j-\hat{\vec{x}}_{(j)})\right\}^{1/2}\\ &\times\frac{1}{n^2}\left\{(\vec{x}_j-\hat{\vec{x}}_{(j)})^{\mathrm T}R_n(\tilde\theta_k-\epsilon)\sum_{p=1,p\neq j}^{n}(\vec{x}_p-\hat{\vec{x}}_{(p)})(\vec{x}_p-\hat{\vec{x}}_{(p)})^{\mathrm T}|\tilde\theta_p(\epsilon)|^{-2}R_n(\tilde\theta_k-\epsilon)^*(\vec{x}_j-\hat{\vec{x}}_{(j)})\right\}^{1/2}\\ &\times\max_{j\neq k}|\tilde\theta_j(\Re\tilde\theta_k,...)-\tilde\theta_j(\Re\tilde\theta_k-\epsilon,...)|+\frac{\rho_{\mathrm{magic}}^2}{\alpha^2[1-\rho_{\mathrm{magic}}\alpha^{-1}-\epsilon]^2}|\epsilon|.\end{aligned}\tag{29.2}$$

Now using the inequality $\vec{x}^*[I\alpha+B+\mathrm{i}C]^{-1}B[I\alpha+B-\mathrm{i}C]^{-1}\vec{x}\le\alpha^{-1}\vec{x}^*\vec{x}$, where

$$B=n^{-1}\sum_{p=1,p\neq j}^{n}(\vec{x}_p-\hat{\vec{x}}_{(p)})(\vec{x}_p-\hat{\vec{x}}_{(p)})^{\mathrm T}\Re\frac{1}{\tilde\theta_p},C=Ix+n^{-1}\sum_{p=1,p\neq j}^{n}(\vec{x}_p-\hat{\vec{x}}_{(p)})(\vec{x}_p-\hat{\vec{x}}_{(p)})^{\mathrm T}\Im\frac{1}{\tilde\theta_p}$$

and the equality $\Re[\tilde\theta_j]^{-1}=\Re\tilde\theta_j|\tilde\theta_j|^{-2}$ we have

$$\begin{aligned}\max_{j\neq k}|\tilde\theta_j(\Re\tilde\theta_k,...)-\tilde\theta_j(\Re\tilde\theta_k-\epsilon,...)| &\le\frac{\rho_{\mathrm{magic}}}{\alpha[1-\rho_{\mathrm{magic}}\alpha^{-1}]}\max_{j\neq k}|\tilde\theta_j(\Re\tilde\theta_k,...)-\tilde\theta_j(\Re\tilde\theta_k-\epsilon,...)|\\ &+\frac{\rho_{\mathrm{magic}}^2}{\alpha^2[1-\rho_{\mathrm{magic}}\alpha^{-1}-\epsilon]^2}|\epsilon|\end{aligned}\tag{29.3}$$

and since $\alpha=\rho_{\mathrm{magic}}+\sqrt{\rho_{\mathrm{magic}}c^{-1}}, c<\min\{1,\rho_{\mathrm{magic}}^{-1}\}$, it follows that

$$\frac{\rho_{\mathrm{magic}}}{\alpha}\frac{1}{(1-\rho_{\mathrm{magic}}\alpha^{-1})}<1$$

and using (29.2) and (29.3) we obtain that the parameters $\tilde\theta_j, j\neq k$ of the function

$$F(\tilde\theta_k,...):=\frac{1}{n}(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm T}\left\{I_{m_n}(\alpha+\mathrm{i}x)+\frac{1}{n}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\hat{\vec{x}}_{(k)})(\vec{x}_j-\hat{\vec{x}}_{(k)})^{\mathrm T}\tilde\theta_j^{-1}\right\}^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})$$

are continuous along this parameter $\Re\tilde{\theta}_k \geq 0$ and are bounded by one due to the choice of the variable $\alpha$. Therefore, these two graphs $y = \Re\tilde{\theta}_k, \Re\tilde{\theta}_k \geq 0$ and $y = F(\Re\tilde{\theta}_k, ...)$ will intersected. Then there exists a solution for this component $\Re\tilde{\theta}_k \geq 0$ of the equation $K_{100}$ at any values of the other components when $\Re\tilde{\theta}_k, \Re\tilde{\theta}_j \geq 0, j \neq k$. The same solution exists for the imaginary part of the component $\Im\tilde{\theta}_k$. But this contradicts to our assumption that this solution does not exist and we obtain that there exists a solution for all other entries $\tilde{\theta}_j$. So, there exists a solution of the system (28.2). Thus, the Lemma 29.2 is proved. □

**Theorem 29.3.** *Let the independent observations $\vec{x}_1, ..., \vec{x}_n$ of the $m_n$-dimensional random vector $\vec{\xi}$, be given, for any $n = 1, 2, ...$ $\mathbf{E}\,\vec{\xi}_k = \vec{a}, k = 1, ..., n$, $R_{m_n} = \mathbf{E}\,(\vec{x}_k - \vec{a})(\vec{x}_k - \vec{a})^{\,*}, k = 1, ..., n$,*

$$\max_{k=1,\dots,n} n^{-1}\vec{x}_k^{\mathrm{T}}\vec{x}_k \leq \rho_{\mathrm{magic}}, \tag{29.4}$$

$$\alpha = \rho_{\mathrm{magic}} + \sqrt{\rho_{\mathrm{magic}} c^{-1}}, c < \min\{1, \rho_{\mathrm{magic}}^{-1}\}, \tag{29.5}$$

*for a certain $\delta > 0$*

$$\lim_{\substack{n,m\to\infty;\\ mn^{-1}\to\gamma}} \max_{\vec{q}:\,\vec{q}^{\,*}\vec{q}\leq 1} \max_{k=1,\dots,m} \mathbf{E}\,|(\vec{x}_k - \vec{a})^{\,*}\vec{q}\,|^{4+\delta} < \infty. \tag{29.6}$$

*Then for any $s > 0$*

$$\lim_{L\to\infty} \operatorname{l.i.m.}_{\substack{n,m_n\to\infty;\\ m_n n^{-1}\to\gamma}} \left\{ \left[ \frac{1}{2\pi m_n} \int_{-L}^{L} e^{-\mathrm{i}xs} \operatorname{Tr} G_{55}(\alpha + \mathrm{i}x)\mathrm{d}x \right] e^{s\alpha} - \int_0^\infty e^{-ys}\mathrm{d}\mu_{m_n}(y) \right\} = 0,$$

*where*

$$\mu_{m_n}(u) = (m_n)^{-1} \sum_{j=1}^{m_n} \chi\{\lambda_j(R_{m_n}) < u\}$$

*is the normalized spectral function and $\lambda_j(R_{m_n})$ are the eigenvalues of the matrix $R_{m_n}$.*

*There exists the unique solution $\tilde{\theta}_k, k = 1, ..., n$ of the canonical equation $K_{100}$ with the non negative parts $\Re\tilde{\theta}_k \geq c > 0, k = 1, ..., n$.*

**Remark 29.4.** *Very often we do not need the spectral function of the covariance matrix $R_{m_n}$, but the normalized trace of the inverse regularized matrix $(m_n)^{-1}\operatorname{Tr}[I_{m_n}\varepsilon + R_{m_n}]^{-1}, \varepsilon > 0$. In this case, the statement of the Theorem 29.3 will look like this*

$$\lim_{M\to\infty} \lim_{L\to\infty} \lim_{\substack{n,m_n\to\infty;\\ m_n n^{-1}\to\gamma}} \frac{1}{m_n}\mathbf{E}\left| \int_0^M \left\{ \frac{1}{2\pi} \int_{-L}^{L} e^{-\mathrm{i}xs} \operatorname{Tr} G_{55}(\alpha + \mathrm{i}x)\mathrm{d}x \right\} e^{s\alpha - \varepsilon s}\mathrm{d}s - \operatorname{Tr}[I_{m_n}\varepsilon + R_{m_n}]^{-1} \right| = 0,$$

*where $\varepsilon > 0$,*

$$G_{55}(\alpha + \mathrm{i}x) = \left\{ I_{m_n}(\alpha + \mathrm{i}x) + n^{-1} \sum_{k=1}^{n} (\vec{x}_k - \hat{\vec{x}}_{(k)})(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}}\theta_k^{-1} \right\}^{-1},$$

*where $\alpha > 0$ is a certain constant, $x$ is an arbitrary parameter, the random variables $\theta_k, k = 1, ..., n$ are satisfied the system of equations (28.2).*

*Proof.* Proof of Theorem 29.3. We start the proof with the main lemma:

**Lemma 29.5.** *Under the conditions of Theorem 29.3 there exists the unique solution $\tilde{\theta}_k, \Re\tilde{\theta}_k > 1 - \rho_{\mathrm{magic}}\alpha^{-1}, k = 1, ..., n$ of the system of canonical equations (28.2).*

*Proof.* We have already proved the existence of solutions $\tilde\theta_k, \Re\tilde\theta_k \geq 0, k=1,...,n$ of the equations in the Lemma 29.2. Let's assume that there are two different solutions $\tilde\theta_k^{(1)}, k=1,...,n$ and $\tilde\theta_k^{(2)}, k=1,...,n$.

Denote

$$\hat R_{m_n}^{(k,1)} = \left\{ I_{m_n}(\alpha + \mathrm{i}x) + n^{-1}\sum_{j=1,j\neq k}^{n} (\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}[\tilde\theta_j^{(1)}]^{-1}\right\}^{-1},$$

$$\hat R_{m_n}^{(k,2)} = \left\{ I_{m_n}\alpha + \mathrm{i}x) + n^{-1}\sum_{j=1,j\neq k}^{n} (\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}[\tilde\theta_j^{(2)}]^{-1}\right\}^{-1},$$

$$G_{m_n}^{(k)} = n^{-1}\sum_{j=1,j\neq k}^{n} (\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}([\tilde\theta_j^{(1)}]^{-1} - [\tilde\theta_j^{(2)}]^{-1}).$$

Then we get since $\Re\tilde\theta_k^{(1)} > 1-\rho_{\text{magic}}\alpha^{-1}, \Re\tilde\theta_k^{(2)} > 1-\rho_{\text{magic}}\alpha^{-1}$

$$\begin{aligned}
\max_{k=1,\ldots,n} |\tilde\theta_k^{(1)} - \tilde\theta_k^{(2)}| &= \max_{k=1,\ldots,n} |n^{-1}(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}[\hat R_{m_n}^{(k,1)} - \hat R_{m_n}^{(k,2)}](\vec x_k - \hat{\vec x}_{(k)})| \\
&= \max_{k=1,\ldots,n} |n^{-1}(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,1)} G_{m_n}^{(k)}\hat R_{m_n}^{(k,2)}(\vec x_k - \hat{\vec x}_{(k)})| \\
&= \max_{k=1,\ldots,n} \Bigg| n^{-2}\sum_{j=1,j\neq k}^{n} (\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,1)}(\vec x_j - \hat{\vec x})(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,2)}(\vec x_j - \hat{\vec x}_{(k)}) \\
&\times([\tilde\theta_j^{(1)}]^{-1} - [\tilde\theta_j^{(2)}]^{-1})\Bigg| \\
&\leq \frac{1}{n^2}\max_{k=1,\ldots,n}\sum_{j=1,j\neq k}^{n}\Big|(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,1)}(\vec x_j - \hat{\vec x})[\tilde\theta_j^{(1)}]^{-1}(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,2)}(\vec x_j - \hat{\vec x}_{(k)})[\tilde\theta_j^{(2)}]^{-1} \\
&\times \max_{k=1,\ldots,n}|\tilde\theta_k^{(1)} - [\tilde\theta_k^{(2)}|.
\end{aligned} \tag{29.7}$$

Again using inequalities $\Re\tilde\theta_k^{(1)} > 1-\rho_{\text{magic}}\alpha^{-1}, \Re\tilde\theta_k^{(2)} > 1-\rho_{\text{magic}}\alpha^{-1}$ we continue

$$\begin{aligned}
\max_{k=1,\ldots,n} |\tilde\theta_k^{(1)} - \tilde\theta_k^{(2)}| &\leq \max_{k=1,\ldots,n}\left\{(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,1)}\sum_{j=1,j\neq k}^{n}(\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}|\tilde\theta_j^{(1)}|^{-2}\hat R_{m_n}^{(k,1)*}(\vec x_k - \hat{\vec x}_{(k)})\right\}^{1/2} \\
&\times\frac{1}{n^2}\left\{(\vec x_k - \hat{\vec x}_{(k)})^{\mathrm T}\hat R_{m_n}^{(k,2)}\sum_{j=1,j\neq k}^{n}(\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}|\tilde\theta_j^{(2)}|^{-2}\hat R_{m_n}^{(k,2)*}(\vec x_k - \hat{\vec x}_{(k)})\right\}^{1/2} \\
&\times \max_{k=1,\ldots,n}|\tilde\theta_k^{(1)} - \tilde\theta_k^{|}.
\end{aligned} \tag{29.8}$$

Now using the inequality $\vec x^*[I\alpha + B + \mathrm{i}C]^{-1}B[I\alpha + B - \mathrm{i}C]^{-1}\vec x \leq \alpha^{-1}\vec x^*\vec x$, where

$$B = n^{-1}\sum_{j=1,j\neq k}^{n}(\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}\Re\frac{1}{\tilde\theta_j^{(1)}}, C = Ix + n^{-1}\sum_{j=1,j\neq k}^{n}(\vec x_j - \hat{\vec x}_{(k)})(\vec x_j - \hat{\vec x}_{(k)})^{\mathrm T}\Im\frac{1}{\tilde\theta_j^{(1)}}$$

and equality $\Re[\tilde\theta_j^{(1)}]^{-1} = \Re\tilde\theta_j^{(1)}|\tilde\theta_j^{(1)}|^{-2}$ we have from the inequalities (29.7) and (29.8) that

$$\max_{k=1,\ldots,n}|\tilde\theta_k^{(1)} - \tilde\theta_k^{(2)}| \leq \frac{\rho_{\text{magic}}}{\alpha}\frac{1}{(1-\rho_{\text{magic}}\alpha^{-1})}\max_{k=1,\ldots,n}|\tilde\theta_k^{(1)} - \tilde\theta_k^{(2)}|$$

and since $\alpha = \rho_{\text{magic}} + \sqrt{\rho_{\text{magic}}c^{-1}}, c < \min\{1, \rho_{\text{magic}}^{-1}\}$, it follows that

$$\frac{\rho_{\text{magic}}}{\alpha}\frac{1}{(1-\rho_{\text{magic}}\alpha^{-1})}<1$$

and we obtain the uniqueness of the solution of stochastic canonical equation $K_{100}$. □

The solution of the canonical system of equations $K_{100}$ is very complicated, it depends on the matrix $\hat{R}_{m_n}$, and therefore they cannot be used to prove the consistency of the estimator $G_{55}$. Therefore, we have considered the accompanying system of canonical equation $K_{101}$ for random variables $\theta_k, k=1,...,n$, which will already be independent:

$$\theta_k + n^{-1}(\vec{x}_k-\vec{a}\ )^{\mathrm{T}}\mathbf{E}\,P_{m_n}^{(k)}(\vec{x}_k-\vec{a}\ )=1, k=1,..,n, \tag{29.9}$$

where

$$P_{m_n}^{(k)}=\left\{I_{m_n}[\alpha+\mathrm{i}x]+\frac{1}{n}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\vec{a}\ )(\vec{x}_j-\vec{a}\ )^{\mathrm{T}}\theta_j^{-1}\right\}^{-1}.$$

Similarly, we prove the following statement

**Lemma 29.6.** *Under the conditions of Theorem 29.3 there exists a unique solution $\theta_k, \Re\theta_k>c>0, k=1,...,n$ of the system of stochastic canonical equations (29.9).*

# 30 Self-averaging of random quadratic forms

As we see, the vectors $(\vec{x}_j-\vec{a})\theta_j^{-1/2}, j=1,...,n$ are stochastically independent, we can prove the following lemma:

**Lemma 30.1.** *Let conditions (29.4)–(29.6) be satisfied. Then*

$$\max_x \max_{k=1,\ldots,n}\mathbf{E}\left[n^{-1}(\vec{x}_k-\vec{a}\ )^*\left(P_{m_n}^{(k)}-\mathbf{E}P_{m_n}^{(k)}\right)(\vec{x}_k-\vec{a}\ )\right]^2\le cn^{-1}.$$

*Proof.* Let $\sigma_{s,k}$ be the smallest $\sigma$-algebra generated by the random vectors $\vec{x}_l,\ l=s+1,\ldots,n,k$. By using the method of martingale differences(see section 14), we get

$$\begin{aligned}
&\max_x \max_{k=1,\ldots,n}\mathbf{E}\left|n^{-1}(\vec{x}_k-\vec{a})^*\left(P_{m_n}^{(k)}-\mathbf{E}P_{m_n}^{(k)}\right)(\vec{x}_k-\vec{a})\right|^2\\
&=\frac{1}{n^2}\max_x \max_{k=1,\ldots,n}\mathbf{E}\left|\sum_{s\neq k;\, s=0}^{n-1}(\vec{x}_k-\vec{a})^*\left(\mathbf{E}\left[P_{m_n}^{(k)}\middle|\sigma_{s,k}\right]-\mathbf{E}\left[P_{m_n}^{(k)}\middle|\sigma_{s+1,k}\right]\right)(\vec{x}_k-\vec{a})\right|^2\\
&\le\frac{cn}{n^2}\max_x\max_{k\neq s}\left|\mathbf{E}\frac{(\vec{x}_k-\vec{a})^*P_{m_n}^{(k,s)}(\vec{x}_s-\vec{a})(\vec{x}_s-\vec{a})^*P_{m_n}^{(k,s)}(\vec{x}_k-\vec{a})}{n\left[\theta_s+n^{-1}(\vec{x}_s-\vec{a})^*P_{m_n}^{(k,s)}(\vec{x}_s-\vec{a})\right]}\right|^2,
\end{aligned}$$

where

$$P_{m_n}^{(k,s)}(z)=\left\{I_{m_n}[\alpha+\mathrm{i}x]+\frac{1}{n}\sum_{j=1,j\neq k,s}^{n}(\vec{x}_j-\vec{a})(\vec{x}_j-\vec{a})^*\theta_j^{-1}\right\}^{-1}.$$

Then since $\Re\theta_s\ge c>0$, we have

$$\max_x \max_{k=1,\dots,n} \mathbf{E}\left|n^{-1}(\vec{x}_k-\vec{a})\ ^*\left(P_{m_n}^{(k)}-\mathbf{E}P_{m_n}^{(k)}\right)(\vec{x}_k-\vec{a})\right|^2$$
$$\le c_1 n^{-1}\max_{k\neq s,k,s=1,\dots,n}\mathbf{E}\left|n^{-1/2}(\vec{x}_k-\vec{a})\ ^*P_{m_n}^{(k,s)}(\vec{x}_s-\vec{a})\right|^4$$
$$\le c_3 n^{-1}\max_{\vec{q}:\ \vec{q}\ ^*\vec{q}\le 1}\max_{k=1,\dots,n}\mathbf{E}\,|(\vec{x}_k-\vec{a})\ ^*\vec{q}|^4\le c_4 n^{-1}.$$

Lemma 30.1 is proved.

In exactly the same way we prove the following statement, which is the main result of the REFORM method

**Lemma 30.2.** *Let conditions (29.4)–(29.6) be satisfied. Then for any $x,\alpha>0$*

$$\max_x \mathbf{E}\,|n^{-1}\mathrm{Tr}\,P_{m_n}^{(k)}-\mathbf{E}\,n^{-1}\mathrm{Tr}\,P_{m_n}^{(k)}|\le cn^{-1}.$$

Denote

$$P_{m_n}=\left\{I_{m_n}[\alpha+\mathrm{i}x]+\frac{1}{n}\sum_{k=1}^{n}(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^*\theta_k^{-1}\right\}^{-1},$$
$$P_{m_n}^{(k)}=\left\{I_{m_n}[\alpha+\mathrm{i}x]+\frac{1}{n}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\vec{a})(\vec{x}_j-\vec{a})^*\theta_j^{-1}\right\}^{-1},$$
$$Q_{m_n}=\left\{I_{m_n}[\alpha+\mathrm{i}x]+\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^*\right\}^{-1}=\{I_{m_n}[\alpha+\mathrm{i}x]+R_{m_n}\}^{-1}$$
$$\rho_k=n^{-1}(\vec{x}_k-\vec{a})\ ^*P_{m_n}^{(k)}(\vec{x}_k-\vec{a})-n^{-1}(\vec{x}_k-\vec{a})\ ^*\mathbf{E}\,P_{m_n}^{(k)}(\vec{x}_k-\vec{a})|. \tag{30.1}$$

Of course

$$Q_{m_n}=\left\{I_{m_n}[\alpha+\mathrm{i}x]+\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\left\{\frac{(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^*}{\theta_k+(\vec{x}_k-\vec{a})^*\{\mathbf{E}\,P_{m_n}^{(k)}(z)\}(\vec{x}_k-\vec{a})}\right\}\right\}^{-1},$$

Now we move on to our main auxiliary statement. Similarly, since the vectors $(\vec{x}_j-\vec{a})\theta_j^{-1/2}, j=1,\dots,n$ are stochastically independent, we can prove the lemma

**Lemma 30.3.** *Let the conditions (29.4)–(29.6) be satisfied. Then*

$$\lim_{n\to\infty}\frac{1}{n}\max_x\left|\mathrm{Tr}\,(Q_{m_n}-\mathbf{E}\,P_{m_n})\right|=0.$$

*Proof.* After some transforms we arrive at the following equation using Lemma 30.1

$$\frac{1}{n}\mathrm{Tr}\,[Q_{m_n}-\mathbf{E}\,P_{m_n}]=\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,[Q_{m_n}-P_{m_n}]+\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,[Q_{m_n}-P_{m_n}]$$
$$=\frac{1}{n}\mathrm{Tr}\,Q_{m_n}\mathbf{E}\left(\frac{1}{n}\sum_{k=1}^{n}(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}\theta_k^{-1}\right.$$
$$\left.-\frac{1}{n}\sum_{k=1}^{n}\left\{\mathbf{E}\,\frac{(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}\{\mathbf{E}\,P_{m_n}^{(k)}(z)\}(\vec{x}_k-\vec{a})}\right\}\right)P_{m_n}$$
$$=\frac{1}{n^2}\mathrm{Tr}\,Q_{m_n}\left(\mathbf{E}\sum_{k=1}^{n}\frac{\rho_k(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}}{[\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}P_{m_n}^{(k)}(\vec{x}_k-\vec{a})][\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}\{\mathbf{E}\,P_{m_n}^{(k)}\}(\vec{x}_k-\vec{a})]}P_{m_n}^{(k)}\right.$$
$$+\sum_{k=1}^{n}\left\{\mathbf{E}\,\frac{(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}\{\mathbf{E}\,P_{m_n}^{(k)}\}(\vec{x}_k-\vec{a})}\right\}P_{m_n}^{(k)}$$
$$\left.-\sum_{k=1}^{n}\left\{\mathbf{E}\,\frac{(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}\{\mathbf{E}\,P_{m_n}^{(k)}\}(\vec{x}_k-\vec{a})}\right\}\mathbf{E}\,P_{m_n}\right)$$
$$=-n^{-2}\mathrm{Tr}\,Q_{m_n}\sum_{k=1}^{n}\mathbf{E}\left\{\frac{(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}\{\mathbf{E}\,P_{m_n}^{(k)}\}(\vec{x}_k-\vec{a})}\right\}$$
$$\times[\mathbf{E}\,P_{m_n}-\mathbf{E}\,P_{m_n}^{(k)}]+\delta_n$$
$$=-n^{-3}\mathrm{Tr}\,Q_{m_n}\sum_{k=1}^{n}\mathbf{E}\left\{\frac{(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}\{\mathbf{E}\,P_{m_n}^{(k)}\}(\vec{x}_k-\vec{a})}\right\}$$
$$\times\mathbf{E}\,\frac{P_{m_n}^{(k)}(\vec{x}_k-\vec{a})(\vec{x}_k-\vec{a})^{\,*}P_{m_n}^{(k)}}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}P_{m_n}^{(k)}(\vec{x}_k-\vec{a})}+\varepsilon_n$$
$$=-\frac{1}{n^3}\mathbf{E}\sum_{k=1}^{n}\frac{[(\vec{x}_k-\vec{a})^{*}P_{m_n}^{(k)}(\vec{y}_k-\vec{a})]}{\theta_k+n^{-1}(\vec{x}_k-\vec{a})^{\,*}P_{m_n}^{(k)}(\vec{x}_k-\vec{a})}\frac{[(\vec{y}_k-\vec{a})^{\,*}P_{m_n}^{(k)}Q_{m_n}(\vec{x}_k-\vec{a})]}{\theta_k+n^{-1}(\vec{y}_k-\vec{a})^{*}P_{m_n}^{(k)}(\vec{y}_k-\vec{a})}+\varepsilon_n,$$

where

$$|\delta_n|\le\sup_x\max_k\mathbf{E}\,n^{-1}|(\vec{x}_k-\vec{a})^{\,*}P_{m_n}^{(k)}Q_{m_n}(\vec{x}_k-\vec{a})||\max_k\mathbf{E}\,|\rho_k|\le\varepsilon_n,\epsilon_n\le cn^{-1/2},$$

the vector $\vec{y}_k$ is stochastically independent of the vector $\vec{x}_k$ and the matrix $P_{m_n}$ and has the same distribution as this vector $\vec{x}_k$. Then we have from this inequality for $\frac{1}{n}\mathrm{Tr}\,[Q_{m_n}-\mathbf{E}\,P_{m_n}]$

$$\max_x|n^{-1}\mathrm{Tr}\,[Q_{m_n}-\mathbf{E}\,P_{m_n}]|\le cn^{-1/2}+cn^{-1}\max_{\vec{q}:\,\vec{q}^{\,*}\vec{q}\le 1}\max_{k=1,\dots,m}\mathbf{E}\,[(\vec{x}_k-\vec{a})^{\,*}\vec{q}\ \,]^2\le cn^{-1/2}.$$

□

Now we can replace the vector $\vec{a}$ by the empirical mean $\hat{\vec{x}}_{(k)}$ by virtue of the formulas for perturbations of random matrices, and we have the following result:

**Lemma 30.4.** *Under the conditions of Theorem 29.3*

$$\max_x \mathrm{l.i.m.}_{\substack{n,m_n\to\infty;\\ m_n n^{-1}\to\gamma}}|n^{-1}\mathrm{Tr}\,\Theta_{m_n}-n^{-1}\mathrm{Tr}\,P_{m_n}]^{-1}|=0,$$

*and*

$$\max_x \mathrm{l.i.m.}_{\substack{n,m_n\to\infty;\\ m_n n^{-1}\to\gamma}}|n^{-1}\mathrm{Tr}\,\Theta_{m_n}-n^{-1}\mathrm{Tr}\,\Theta_{m_n}]^{-1}|=0,$$

*where*

$$\Theta_{m_n}=\left\{I_{m_n}(\alpha+\mathrm{i}x)+\frac{1}{n}\sum_{k=1}^{n}(\vec{x}_k-\hat{\vec{x}}_{(k)})(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\theta_k^{-1}\right\}^{=1},$$

*Proof.* We know that $\theta_k + n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}} \mathbf{E}\, P^{(k)}_{m_n}(\vec{x}_k - \vec{a}) = 1, k = 1, .., n.$ and we now need to replace the vector $\vec{a}$ in this system of equations with the empirical mean $\hat{\vec{x}}$. This is easy to do because

$$\frac{1}{n}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \vec{a})(\vec{x}_j - \vec{a})\ ^{*}\theta_j^{-1} = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4,$$

where

$$\Gamma_1 = \frac{1}{n}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})(\vec{x}_j - \hat{\vec{x}}_{(k)})^{*}\theta_j^{-1}, \Gamma_2 = (\vec{a} - \hat{\vec{x}}_{(k)})(\vec{a} - \hat{\vec{x}}_{(k)})\ ^{*}\frac{1}{n}\sum_{j=1, j\neq k}^{n}\theta_j^{-1},$$

$$\Gamma_3 = -(\vec{a} - \hat{\vec{x}}_{(k)})\frac{1}{n}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})^{*}\theta_j^{-1}, \Gamma_4 = \frac{1}{n}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})^{*}\theta_j^{-1}(\vec{a} - \hat{\vec{x}}_{(k)}).$$

Then $P^{(k)}_{m_n} := P^{(k)}_{(4)}$, where

$$P^{(k)}_{(s)} = \left\{ I_{m_n}[\alpha + \mathrm{i}x] + \sum_{p=1,\ldots,s}\Gamma_p \right\}^{-1}, \hat{\vec{x}}_{(k)} = \frac{1}{n}\sum_{j=1, j\neq k}^{n}\vec{x}_j, s = 1, ..., 4.$$

Let $\vec{b}_{(k)} = n^{-1}\sum_{j=1, j\neq k}^{n}(\vec{x}_j - \hat{\vec{x}}_{(k)})^{*}\theta_j^{-1}$. We will take into account the following simple inequalities

$$\frac{1}{n}\max_{x}\mathbf{E}\,|\hat{\vec{x}}^{*}_{(k)}Q_{m_n}\hat{\vec{x}}_{(k)}| \leq \frac{c}{n}, \quad \frac{1}{n}\mathbf{E}\,|\hat{\vec{x}}^{*}_{(k)}\hat{\vec{c}}| \leq \frac{c}{n^{1/2}},$$

where $\vec{c}\ ^{*}\vec{c} \leq c$ and $Q_{m_n}$ is any positive definite Hermitian matrix with bounded eigenvalues, and

$$\mathbf{E}\,[\vec{b}^{*}_{(k)}\vec{b}_{(k)}] \leq cn^{-2}\sum_{j=1, j\neq k}^{n}\mathbf{E}\,(\vec{x}_j - \vec{a})^{*}(\vec{x}_j - \vec{a}) \leq cn^{-1}.$$

Obviously

$$\frac{1}{n}[\mathrm{Tr}\, P^{(k)}_{(4)} - \mathrm{Tr}\, P^{(k)}_{(1)}] = -\sum_{s=2}^{4}\frac{1}{n}\mathrm{Tr}\, P^{(k)}_{(4)}\Gamma_s P^{(k)}_{(1)}$$

and we have the following inequalities

$$\begin{aligned}\sup_x \mathbf{E}\left|\frac{1}{n}\mathrm{Tr}\, P^{(k)}_{(4)}\Gamma_3 P^{(k)}_{(1)}\right| &\leq \sup_x \frac{1}{n}[\mathbf{E}\,(\vec{a} - \hat{\vec{x}}_{(k)})^{*}P^{(k)}_{(4)}P^{(k)*}_{(4)}(\vec{a} - \hat{\vec{x}}_{(k)})]^{1/2}[\mathbf{E}\,\vec{b}^{*}_{(k)}P^{(k)}_{(1)}P^{(k)*}_{(1)}\vec{b}_{(k)}]^{1/2} \\ &\leq \frac{c}{n}[\mathbf{E}\,(\vec{a} - \hat{\vec{x}}_{(k)})^{*}(\vec{a} - \hat{\vec{x}}_{(k)})]^{1/2}[\mathbf{E}\,\vec{b}^{*}_{(k)}\vec{b}_{(k)}]^{1/2} \leq \frac{c}{n}.\end{aligned}$$

Similarly we get

$$\sup_x \mathbf{E}\left|n^{-1}\mathrm{Tr}\, P^{(k)}_{(4)}\Gamma_4 P^{(k)}_{(1)}\right| \leq \frac{c}{n}, \sup_x \mathbf{E}\,|n^{-1}\mathrm{Tr}\, P^{(k)}_{(4)}\Gamma_2 P^{(k)}_{(1)}| \leq \frac{c}{n}.$$

Therefore, under the conditions of the Theorem 29.3

$$\lim_{\substack{n, m_n \to \infty; \\ m_n n^{-1} \to \gamma}}\frac{1}{n}\max_x \mathbf{E}\,|\mathrm{Tr}\, P^{(k)}_{(4)} - \mathrm{Tr}\, P^{(k)}_{(1)}| = 0.$$

Then we complete the proof.

□

The solution of the canonical system of equations $K_{100}$ is very complicated, it depends on the matrix $\hat{R}_{m_n}$, and therefore they cannot be used to prove the consistency of the estimator $G_{55}$. Therefore, we have considered the accompanying system of equations (29.9) for random variables $\theta_k, k = 1, ..., n$, which will already be independent: We will carry out the same procedure for replacing the vector $\vec{a}$ with the empirical mean $\hat{\vec{x}}$ in the system of canonical equations $K_{100}$, in addition, we must replace the matrix $\mathbf{E}\, P_{m_n}^{(k)}$ with the matrix $P_{m_n}^{(k)}$, but in this case we will need to estimate the absolute moments of random variables of the order $2+\delta$.

**Lemma 30.5.** *Under the conditions of the Theorem 29.3 the solutions $\theta_k, k=1,...,n$ satisfy the system of canonical equations*

$$\theta_k + n^{-1}(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}} P_{(1)}^{(k)} (\vec{x}_k - \hat{\vec{x}}_{(k)}) = 1 + \epsilon_k, k = 1,..,n, \tag{30.2}$$

*and for a certain $\delta > 0$*

$$\lim_{n\to\infty} \max_{x} \mathbf{E} \max_{k=1,...,n} |\varepsilon_k|^{2+\delta} = 0.$$

*Proof.* As in the previous lemma we have

$$n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}} \mathbf{E}\, P_{(4)}^{(k)}(\vec{x}_k - \vec{a}) - n^{-1}(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}} P_{(1)}^{(k)}(\vec{x}_k - \hat{\vec{x}}_{(k)}) = L_1^{(k)} + L_2^{(k)} + L_3^{(k)} + L_4^{(k)} + L_5^{(k)}, \tag{30.3}$$

where

$$\begin{aligned} L_1^{(k)} &= n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}}[\mathbf{E}\, P_{(4)}^{(k)} - P_{(4)}^{(k)}](\vec{x}_k - \vec{a}), L_2^{(k)} = -n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}}[P_{(4)}^{(k)}\Gamma_4 P_{(3)}^{(k)}](\vec{x}_k - \vec{a}), \\ L_3^{(k)} &= n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}}[P_{(3)}^{(k)}\Gamma_3 P_{(2)}^{(k)}](\vec{x}_k - \vec{a}), L_4^{(k)} = n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}}[P_{(2)}^{(k)}\Gamma_2 P_{(1)}^{(k)}](\vec{x}_k - \vec{a}), \\ L_5^{(k)} &= 2n^{-1}(\vec{x}_k - \hat{\vec{x}}_{(k)})^{\mathrm{T}} P_{(1)}^{(k)}(\hat{\vec{x}}_{(k)} - \vec{a}) + n^{-1}(\vec{a} - \hat{\vec{x}}_{(k)})^{\mathrm{T}} P_{(1)}^{(k)}(\vec{a} - \hat{\vec{x}}_{(k)}). \end{aligned} \tag{30.4}$$

Since the vector $\vec{x}_k$ does not depend on the matrix $P_{(1)}^{(k)}$ then we will take into account the inequality (14.4) for the moments of the sum of martingale-differences and we obtain the following simple inequalities for any $\delta > 0$

$$\begin{aligned} \mathbf{E}\,|L_1^{(k)}|^{2+\delta} &= \mathbf{E}\,|n^{-1}(\vec{x}_k - \vec{a})^{\mathrm{T}}[\sum_{j=1,..,n-1, j\neq k} [\mathbf{E}_{\,j-1} - \mathbf{E}_{\,j}] P_{(4)}^{(k)}(\vec{x}_k - \vec{a})|^{2+\delta} \\ &\leq \frac{cn^{\delta/2} n}{n^{-2-\delta}} \max_{\vec{q}:\ \vec{q}\ ^*\vec{q}\leq 1} \max_{k=1,...,m} \mathbf{E}\,[(\vec{x}_k - \vec{a})\ ^*\vec{q}\ \ ]^{4+2\delta} \leq cn^{-1-\delta/2}, \end{aligned} \tag{30.5}$$

where $\mathbf{E}_{\,j}$ is the conditional expectation under fixed random vectors $\vec{x}_p, p = j,..,n, j \neq k$

$$\mathbf{E}\,|L_2^{(k)}|^{2+\delta} \leq cn^{-2-\delta/2} \max_{\vec{q}:\ \vec{q}\ ^*\vec{q}\leq 1} \max_{k=1,...,m} \mathbf{E}\,[(\vec{x}_k - \vec{a})\ ^*\vec{q}\ \ ]^{4+\delta} \leq cn^{-1-\delta/2},$$

$$\begin{aligned} \mathbf{E}\,|L_3^{(k)}|^{2+\delta} &\leq \frac{c}{n^{\delta}} \sqrt{\mathbf{E}\,|(\vec{x}_k - \vec{a})\ ^*(\vec{a} - \hat{\vec{x}}_{(k)})|^{4+2\delta}} \sqrt{\mathbf{E}\,|(\vec{x}_k - \vec{a})\ ^*(\vec{x}_k - \hat{\vec{x}}_{(k)})|^{4+2\delta}} \\ &\leq n^{-2-\delta/2} \max_{\vec{q}:\vec{q}\ ^*\vec{q}\leq 1} \max_{k=1,...,n} \mathbf{E}\,[(\vec{x}_k - \vec{a})\ ^*\vec{q}]^{4+\delta} \leq cn^{-1-\delta/2}, \end{aligned} \tag{30.6}$$

Similarly

$$\mathbf{E}\,|L_4^{(k)}|^{2+\delta} \leq n^{-2-\delta/2} \max_{\vec{q}:\vec{q}\ ^*\vec{q}\leq 1} \max_{k=1,...,n} \mathbf{E}\,[(\vec{x}_k - \vec{a})\ ^*\vec{q}]^{4+\delta} \leq cn^{-1-\delta/2},$$

$$\mathbf{E}\,|L_5^{(k)}|^{2+\delta} \leq n^{-2-\delta/2} \max_{\vec{q}:\vec{q}\ ^*\vec{q}\leq 1} \max_{k=1,...,n} \mathbf{E}\,[(\vec{x}_k - \vec{a})\ ^*\vec{q}]^{4+\delta} \leq cn^{-1-\delta/2}.$$

Then

$$\max_{k=1,\dots,n} \mathbf{E}\,|n^{-1}(\vec{x}_k-\vec{a})^{\mathrm{T}}\mathbf{E}\,P_{(4)}^{(k)}(\vec{x}_k-\vec{a})-n^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm{T}}P_{(1)}^{(k)}(\vec{x}_k-\hat{\vec{x}}_{(k)})|^{2+\delta}$$
$$\le c\sum_{s=1,..,5}\max_{k=1,\dots,n}\mathbf{E}\,|L_s^{(k)}|^{2+\delta}$$
$$\le cn^{-1-\delta/2}[c+\max_{\vec{q}:\vec{q}\ {}^*\vec{q}\le 1}\max_{k=1,\dots,n}\mathbf{E}\,[(\vec{x}_k-\vec{a})\ {}^*\vec{q}]^{4+\delta}]\le cn^{-1-\delta/2}. \tag{30.7}$$

Therefore we complete the proof of Lemma 30.5. □

A remarkable feature of the solutions of these two systems of canonical equations (28.2) and (29.9) is that they approach each other as $n\to\infty$. In the same manner, we prove

**Lemma 30.6.** *Under the conditions of the Theorem 29.3*

$$\lim_{\substack{n,m_n\to\infty;\\ m_n n^{-1}\to\gamma}}\max_x \mathbf{E}\max_{k=1,\dots,n}|\tilde{\theta}_k-\theta_k|^{2+\delta}=0.$$

*Proof.* Using Lemma 30.5 we have obtained two system of equations

$$\theta_k+n^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\{I_{m_n}(\alpha+\mathrm{i}x)+n^{-1}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\hat{\vec{x}}_{(k)})(\vec{x}_j-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\theta_j^{-1}\}^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})=1+\epsilon_k, k=1,..,n, \tag{30.8}$$

$$\tilde{\theta}_k+n^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\{I_{m_n}(\alpha+\mathrm{i}x)+n^{-1}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\hat{\vec{x}}_{(k)})(\vec{x}_j-\hat{\vec{x}}_{(k)})^{\mathrm{T}}\tilde{\theta}_j^{-1}\}^{-1}(\vec{x}_k-\hat{\vec{x}}_{(k)})=1, k=1,..,n, \tag{30.9}$$

where $\vec{x}^{(k)}=n^{-1}\sum_{j=1,j\neq k}^{n}\vec{x}_j$ and for a certain $\delta>0$

$$\lim_{n\to\infty}\max_x \mathbf{E}\max_{k=1,\dots,n}|\varepsilon_k|^{2+\delta}=0.$$

Then similarly as in the proof of Lemma 29.5 we obtain

$$\sup_x \mathbf{E}\max_{k=1,\dots,n}|\tilde{\theta}_k-\theta_k|^{2+\delta}\le c_1\sup_x\mathbf{E}\max_{k=1,\dots,n}|\tilde{\theta}_k-\theta_k|^{2+\delta}+\sup_x c\mathbf{E}\max_{k=1,\dots,n}|\varepsilon_k|^{2+\delta}, c_1<1.$$

Lemma 30.6 is proved.

□

We have done all the preparatory work and now we can prove

**Lemma 30.7.** *Under the conditions of Theorem 29.3*

$$\mathrm{l.i.m.}_{\substack{n,m_n\to\infty;\\ m_n n^{-1}\to\gamma}}\sup_x|m_n^{-1}\mathrm{Tr}\,G_{55}[(\alpha+\mathrm{i}x]-m_n^{-1}\mathrm{Tr}\,\{I_{m_n}[\alpha+\mathrm{i}x]+R_{m_n}\}^{-1}=0. \tag{30.10}$$

*Proof.* Using Lemmas 30.1–30.6 we have

$$\begin{aligned} m_n^{-1}\mathrm{Tr}\,G_{55}[\alpha+\mathrm{i}x] \quad & -m_n^{-1}\mathrm{Tr}\,\{I_{m_n}[\alpha+\mathrm{i}x]+R_{m_n}\}^{-1}=m_n^{-1}\mathrm{Tr}\,G_{55}[\alpha+\mathrm{i}x]-m_n^{-1}\mathrm{Tr}\,P_{(1)}^{(k)}\\ & +m_n^{-1}\mathrm{Tr}\,P_{(1)}^{(k)}-m_n^{-1}\mathrm{Tr}\,P_{(4)}^{(k)}+m_n^{-1}\mathrm{Tr}\,P_{(4)}^{(k)}-m_n^{-1}\mathrm{Tr}\,\{I_{m_n}[\alpha+\mathrm{i}x]+R_{m_n}\}^{-1}. \end{aligned}$$

and

$$\begin{aligned}\sup_x \mathbf{E}\,|m_n^{-1}\mathrm{Tr}\, G_{55}[\alpha+\mathrm{i}x] \quad & -m_n^{-1}\mathrm{Tr}\, P_{(1)}^{(k)}| \\ &= \sup_x \mathbf{E}\,|m_n^{-1}\mathrm{Tr}\, G_{55}[\alpha+\mathrm{i}x]\frac{1}{n}\sum_{j=1,j\neq k}^{n}(\vec{x}_j-\hat{\vec{x}}_{(k)})(\vec{x}_j-\hat{\vec{x}}_{(k)})^*[\tilde{\theta}_j^{-1}-\theta_j^{-1}]P_{(1)}^{(k)}| \\ &\leq c\sup_x \mathbf{E}\max_{j=1,\dots,n}|\tilde{\theta}_j-\theta_j|. \end{aligned} \tag{30.11}$$

Using Lemmas 30.1–30.7 and (30.1)–(30.10) we clearly conclude that the proof of the Theorem 29.3 is complete.

# 31 Estimator $G_{56}$. Stochastic canonical equation $K_{102}$

We can easily generalize estimator $G_{55}$ for the matrices of a more general form

$$\frac{1}{n}\sum_{j=1,\dots,n}^{n} c_j\vec{x}_j\vec{x}_j^{\,\mathrm{T}},$$

where $0<c_j<c, j=1,...,n$ are certain non random constants, $\mathbf{E}\,\vec{x}_j=\vec{0}, \mathbf{E}\,\vec{x}_j\vec{x}_j^{\,\mathrm{T}}=R_{m_n}^{(j)}, j=1,..,n.$

In this case this estimator is equal to

$$G_{56}(\alpha+\mathrm{i}x)=\left\{I_{m_n}(\alpha+\mathrm{i}x)+n^{-1}\sum_{k=1}^{n}c_k\vec{x}_k\vec{x}_k^{\,\mathrm{T}}\hat{\theta}_k^{-1}\right\}^{-1},$$

where $\alpha>0$ is a certain constant, $x$ is an arbitrary parameter, the random complex variables $\hat{\theta}_k, \Re\hat{\theta}_k>0, k=1,...,n$ are satisfied the system of stochastic canonical equations $K_{102}$

$$\hat{\theta}_k+\frac{1}{n}c_k\vec{x}_k^{\,\mathrm{T}}\left\{I_{m_n}(\alpha+\mathrm{i}x)+\frac{1}{n}\sum_{j=1,j\neq k}^{n}c_j\vec{x}_j\vec{x}_j^{\,\mathrm{T}}\hat{\theta}_j^{-1}\right\}^{-1}\vec{x}_k=1, k=1,..,n.$$

**Theorem 31.1.** *Let the independent observations $\vec{x}_1,...,\vec{x}_n$ of the $m_n$-dimensional random vector $\vec{\xi}$, be given, for any $n=1,2,...$ $\mathbf{E}\,\vec{\xi}_k=\vec{0}, k=1,...,n,$ $R_{m_n}^{(k)}=\mathbf{E}\,\vec{x}_k\vec{x}_k^{\,\mathrm{T}}, k=1,...,n,$*

$$\max_{k=1,\dots,n} n^{-1}c_k\vec{x}_k^{\mathrm{T}}\vec{x}_k\leq\rho_{\mathrm{magic}},$$

*for a certain $\delta>0$*

$$\lim_{\substack{n,m\to\infty;\\ mn^{-1}\to\gamma}}\ \max_{\vec{q}:\,\vec{q}^{\,\mathrm{T}}\vec{q}\leq 1}\ \max_{k=1,\dots,m}\mathbf{E}\left|\vec{x}_k^{\,\mathrm{T}}\vec{q}\,\right|^{4+\delta}<\infty,$$

$$\lim_{n\to\infty}m_nn^{-1}=\gamma, \alpha=\rho_{\mathrm{magic}}+\sqrt{\rho_{\mathrm{magic}}c^{-1}}, c<\min\{1,\rho_{\mathrm{magic}}^{-1}\}.$$

*Then for any $s>0$ and any $\epsilon>0$*

$$\lim_{M\to\infty}\lim_{L\to\infty}\lim_{\substack{n,m_n\to\infty;\\ m_nn^{-1}\to\gamma}}\frac{1}{2\pi m_n}\mathbf{E}\left|\int_0^M\left\{\int_{-L}^{L}e^{-\mathrm{i}xs}\right.\right.$$
$$\left.\left.\times\mathrm{Tr}\,G_{56}(\alpha+\mathrm{i}x)\mathrm{d}x\right\}\mathrm{d}xe^{s\alpha-\varepsilon s}\mathrm{d}s-\mathrm{Tr}\left[I_{m_n}\varepsilon+\frac{1}{n}\sum_{k=1,\dots,n}c_kR_{m_n}^{(k)}\right]^{-1}\right|=0.$$

*There exists the unique solution $\hat{\theta}_k, k=1,...,n$ of the canonical equation $K_{102}$ with non negative parts $\Re\tilde{\theta}_k\geq 1-\rho_{\mathrm{magic}}\alpha^{-1}\geq c>0, k=1,...,n,$ where $c$ is a positive constant.*

# 32 The MAGIC estimator $G_{57}$ of mean. Stochastic canonical equation $K_{103}$ and accompanying stochastic canonical equation $K_{104}$

This is the main goal of our research. That is, we turn to the age-old problem of estimating a vector $\vec{a}$ by the independent observations $\vec{x}_k, k = 1, ..., N$. Let us immediately note that the most difficult case is when the vector $\vec{a}$ contains many components and it makes no sense to write it in one line, and we prepare these components in the form of a table. But this table is the matrix and we now interpret our observations $\vec{x}_k, k = 1, ..., n$ as the observations $\Xi_n^{(k)} = \{x_{ij}^{(k)}\}, k = 1, ..., n$ on a certain matrix $A_n$.

**Remark 32.1.** *Moreover, in constructing such a matrix $A_n$, we have many possibilities. For example, the matrix $A_n$ of mean values when all entries of the matrix are equal to a, has eigenvalues equal to {na,0,...,0}. And sometimes it is not convenient to use such matrices, but we can multiply all observations $x_{ij}^{(k)}$ by constants $h_{ij}$ chosen in such a way that the matrix $H_n = \{h_{ij}\}$ is orthogonal. Then the eigenvalues of the matrix $\{ah_{ij}^{(k)}\}$ will be bounded by some constant.*

Let us further recall that many problems are related to the statistical estimation of a certain matrices, for example, in the numerical analysis when solving the systems of linear algebraic equations $A_n\vec{y}_n = \vec{b}_n$ or in linear stochastic programming. Moreover, in accordance with MAGIC, as a rule, these problems come down to finding some functions of these matrices, for example, traces of their resolvents $\operatorname{Tr} A_n^*[A_n A_n^* + \epsilon I_n]^{-1}, \epsilon > 0$. Note that the resulting estimator $G_{57}$ has a complex form, but it can significantly reduce the number of necessary observations on the matrix $A_n$.

We move on to the finding estimators of $\operatorname{Tr}[A_n + \mathrm{i}\epsilon I_n]^{-1}$ for symmetrical matrix using canonical equations and observations $\Xi_n^{(j)}, j = 1, ..., n$ of a random matrix $\Xi_n, \mathbf{E}\,\Xi_n = A_n$. We will show how this can be done using an example equation $K_{27}$ and we consider the estimator

$$G_{57} = \frac{1}{n} \sum_{j=1,\dots,n} \Xi_n^{(j)} + \Theta_n(x + \mathrm{i}\alpha), \tag{32.1}$$

and the complex matrix $\Theta_n(x + \mathrm{i}\alpha)$ satisfies the system of canonical equations $K_{103}$

$$\Theta_n(x+\mathrm{i}\alpha) = \frac{1}{n^2} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat{\Xi}_n] R_n(x+\mathrm{i}\alpha) [\Xi_n^{(j)} - \hat{\Xi}_n] \chi \left\{ \max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - A_n]^2 \right\} < \beta_{\text{megic}}^2 \right\}, \tag{32.2}$$

where $\hat{\Xi}_n = n^{-1} \sum_{k=1,\dots,n} \Xi_n^{(k)}$, $\lambda_k(.)$ are the eigenvalues of a matrix,

$$R_n(x + \mathrm{i}\alpha) = \left\{ (x + \mathrm{i}\alpha) I_n + \Theta_n(x + \mathrm{i}\alpha) + \frac{1}{n} \sum_{k=1,\dots,n} \Xi_n^{(k)} \right\}^{-1}.$$

We consider also the auxiliary accompanying system of canonical equations $K_{104}$

$$\tilde{\Theta}_n(x + \mathrm{i}\alpha) = n^{-2} \mathbf{E} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - A_n] \mathbf{E}\, T_n(x + \mathrm{i}\alpha) [\Xi_n^{(j)} - A_n], \tag{32.3}$$

where

$$T_n(x + \mathrm{i}\alpha) = \left\{ (x + \mathrm{i}\alpha) I_n + \tilde{\Theta}_n(x + \mathrm{i}\alpha) + \frac{1}{n} \sum_{k=1,\dots,n} \Xi_n^{(k)} \right\}^{-1}.$$

# 33 Energy conditions of MAGIC. The basic properties and inequalities for solution of the stochastic canonical equations $K_{103}$ and $K_{104}$

Let us introduce the Energy condition of MAGIC

$$p \lim_{n\to\infty} \max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - A_n]^2 \right\} < \beta_{\text{megic}} \tag{33.1}$$

and

$$\alpha^2 > \max\{16\beta_{\text{magic}} c, 8\sqrt{\beta}_{\text{magic}} c\}, c > 1. \tag{33.2}$$

where $\beta_{\text{megic}}$ is a certain constant which plays a decisive role in our theory and that is why we have given it this notation. We also need the following Energy condition of MAGIC

$$\max_{n=1,2,\dots} \max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} \mathbf{E}\, [\Xi_n^{(j)} - A_n]^2 \right\} < \gamma_{\text{megic}} \tag{33.3}$$

and

$$\alpha^2 > \max\{16\gamma_{\text{magic}} c, 8\sqrt{\gamma}_{\text{magic}} c\}, c > 1, \tag{33.4}$$

where $\gamma_{\text{megic}}$ is a certain constant which plays a decisive role in our theory and that is why we have given it this notation.

Denote the solutions of the canonical equations (32.1)–(32.3) as $\Theta_n = \Theta_n^{(1)} + \mathrm{i}\Theta_n^{(2)}$, $\bar\Theta_n = \bar\Theta_n^{(1)} + \mathrm{i}\bar\Theta_n^{(2)}$, where $\Theta_n^{(1)}, \Theta_n^{(2)}, \bar\Theta_n^{(1)}, \bar\Theta_n^{(2)}$ are symmetric real matrices and the classes of complex symmetrical matrices

$$\Upsilon = \left\{ \Theta_n : 2 \max_{k=1,\dots,n} |\lambda_k\{\Theta_n^{(2)}\}| \le \alpha - \sqrt{\alpha^2 - 4\beta_{\text{megic}}} \right\},$$

$$\Pi = \left\{ \Theta_n : 2 \max_{k=1,\dots,n} |\lambda_k\{\bar\Theta_n^{(2)}\}| \le \alpha - \sqrt{\alpha^2 - 4\gamma_{\text{megic}}} \right\}$$

and denote the event

$$\Omega_n = \left\{ \omega : \max_{k=1,\dots,n} \sqrt{\lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - A_n]^2 \right\}} < \beta_{\text{megic}} \right\}.$$

**Lemma 33.1.** *Under the conditions (33.1) and (33.2) the solution $\Theta_n$ of the canonical equation $K_{103}$ satisfies the inequality*

$$\max_{k=1,\dots,n} |\lambda_k\{\Theta_n^{(2)}\}| \chi\{\Omega_n\} \le \frac{\alpha - \sqrt{\alpha^2 - 4\beta_{\text{megic}}}}{2}.$$

*Proof.* Let $\Theta_n^{(2)} = U_n \Lambda_n U_n^*$, where $U_n$ is the orthogonal matrix and $\Lambda_n$ is the diagonal matrix of eigenvalues. Then we have from the equation (32.2)

$$\Lambda_n = -U_n^* n^{-2} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat\Xi_n] U_n F_n(x,\alpha) U_n^* [\Xi_n^{(j)} - \hat\Xi_n] U_n \chi\{\Omega_n\},$$

where $F_n(x,\alpha) = \{[I_n\alpha + \Lambda_n] + D_n\}^{-1}$,

$$D_n = n^{-2} U_n^* \left( I_n x + \sum_{k=1,\dots,n} \Xi_n^{(k)} + \Theta_n^{(1)} \right) \left[ I_n\alpha + \Theta_n^{(2)} \right]^{-1} \left( I_n x + \sum_{k=1,\dots,n} \Xi_n^{(k)} + \Theta_n^{(1)} \right) U_n.$$

Hence

$$\max_{k=1,\dots,n} |\lambda_k| \le \frac{\beta_{\text{megic}}}{\alpha - \max_{k=1,\dots,n} |\lambda_k|}.$$

Therefore, solving this inequality and taking in mind that $\alpha - \max_{k=1,\dots,n} |\lambda_k| > 0$ we arrive at the statement of the Lemma 33.1. □

Similarly we get

**Lemma 33.2.** *Under the conditions (33.3) and (33.4) the solution $\bar{\Theta}_n$ of the canonical equation $K_{104}$ satisfies the inequality*

$$\max_{k=1,\dots,n} |\lambda_k\{\bar{\Theta}_n^{(2)}\}| \le \frac{\alpha - \sqrt{\alpha^2 - 4\gamma_{\text{megic}}}}{2}.$$

**Lemma 33.3.** *Under the conditions (33.1) and (33.2) there exists the solution $\Theta_n, I_n\alpha + \Theta_n^{(2)} > 0$ of the canonical equation $K_{103}$ in the class of complex symmetrical matrices $\Upsilon$.*

*Proof.* Obviously

$$\max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - \hat{X}_n]^2 \right\} \le 2 \max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - A_n]^2 \right\}$$
$$+ 2 \max_{k=1,\dots,n} \lambda_k \left\{ \frac{1}{n} [\hat{X}_n - A_n]^2 \right\} \le 4 \max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - A_n]^2 \right\}.$$

We have under the condition $I_n\alpha + \Theta_n^{(2)} > 0$ that

$$\Theta_n^{(2)} = -n^{-2} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat{\Xi}_n] W_n(\alpha) [\Xi_n^{(j)} - \hat{\Xi}_n] \chi\{\Omega_n\},$$

where

$$W_n(\alpha) = \Bigg\{ I_n\alpha + \Theta_n^{(2)} + \frac{1}{n^2} \Bigg( I_n x$$
$$+ \sum_{k=1,\dots,n} \Xi_n^{(k)} + \Theta_n^{(1)} \Bigg) [I_n\alpha + \Theta_n^{(2)}]^{-1} \Bigg( I_n x + \sum_{k=1,\dots,n} \Xi_n^{(k)} + \Theta_n^{(1)} \Bigg) \Bigg\}^{-1} \chi\{\Omega_n\}.$$

Let's argue backwards and let at least one solution, say $\Im\theta_{pl}^{(2)}$, does not exist and without loss of generality we assume that the other solutions $\theta_{ij}\{\theta_{pl}^{(2)}\}, i, j \ne p, l$ exist. Let $\Theta_n^{(pl)}$ be the matrix whose entry $\theta_{pl}$ equal zero. The other entries of the matrix $\Theta_n^{(pl)}(\Im\theta_{pl})$ will be continues functions of this element $\Im\theta_{pl}^{(2)}$ since for any $\epsilon$

$$\begin{aligned} \Theta_n^{(pl)}(\Im\theta_{pl}) - \Theta_n^{(pl)}(\Im\theta_{pl} - \epsilon) \qquad &= n^{-2} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat{\Xi}_n] T_n(x + \mathrm{i}\alpha, \theta_{pl}) \\ &\times \{\Theta_n^{(pl)}(\Im\theta_{pl}) - \Theta_n^{(pl)}(\Im\theta_{pl} - \epsilon) + H_n(\epsilon)\} \\ &\times T_n(x + \mathrm{i}\alpha, \theta_{pl} - \epsilon) [\Xi_n^{(j)} - \hat{\Xi}_n] \chi\{\Omega_n\}, \end{aligned}$$

where the matrix $T_n(x + \mathrm{i}\alpha, \theta_{pl})$ is defined in (32.3), $H_n(\epsilon) = (h_{ij})$ is a matrix whose all entries are zero except $h_{pl} = \epsilon$ and if we will use spectral decomposition $\Theta_n^{(pl)}(\Im\theta_{pl}) - \Theta_n^{(pl)}(\Im\theta_{pl} - \epsilon)) = U_n \Lambda_n V_n$, where $U_n, V_n$ are Unitary matrices and $\Lambda_n = (\lambda_k \delta_{kl}), \lambda_1 \ge \dots \ge \lambda_n$ is the diagonal matrix of singular eigenvalues we get since $|\lambda_k\{\Theta_n^{(2)}\}| \le 2^{-1}[\alpha - \sqrt{\alpha^2 - 4\beta_{\text{megic}}}]$ and $\alpha^2 = 16c\beta_{\text{megic}}, c > 1$

$$\begin{aligned}
\lambda_1 &= \max_{\vec{x},\vec{y}:\vec{x}^{\;*}\vec{x}\le 1,\vec{y}^{\;*}\vec{y}\le 1} \vec{x}^{\;*}[U_n\Lambda_n V_n + H_n(\epsilon)]\vec{y} \\
&= \max_{\vec{x},\vec{y}:\vec{x}^{\;*}\vec{x}\le 1,\vec{y}^{\;*}\vec{y}\le 1} \vec{x}^{\;*} n^{-2} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat{\Xi}_n] R_n(x+\mathrm{i}\alpha, \Theta_n^{(u)}) U_n \Lambda_n V_n \\
&\times R_n(x+\mathrm{i}\alpha, \Theta_n^{(v)})[\Xi_n^{(j)} - \hat{\Xi}_n]\vec{y}\chi\{\Omega_n\} \\
&\le 4\alpha^{-2}n^{-2} \max_{\vec{x},\vec{y}:\vec{x}^{\;*}\vec{x}\le 1,\vec{y}^{\;*}\vec{y}\le 1} \sum_{j=1,\dots,n} \sqrt{\vec{x}^{\;*}[\Xi_n^{(j)} - \hat{\Xi}_n]^2\vec{x}}\sqrt{\vec{y}^{\;*}[\Xi_n^{(j)} - \hat{\Xi}_n]^2\vec{y}}\ \ \lambda_1\chi\{\Omega_n\} \\
&+4\alpha^{-2}\beta_{\text{magic}}\epsilon \\
&\le 4\alpha^{-2}n^{-2} \max_{\vec{x}:\vec{x}^{\;*}\vec{x}\le 1} \vec{x}^{\;*} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat{\Xi}_n]^2\vec{x}\chi\{\Omega_n\}\ \ \lambda_1 + 4\alpha^{-2}\beta_{\text{magic}}\epsilon \\
&\le 16\alpha^{-2}\beta_{\text{magic}}\lambda_1 + 4\alpha^{-2}\beta_{\text{magic}}\epsilon \le c\lambda_1, c<1.
\end{aligned}$$

Therefore, the entries of the matrix $W_n$ are continuous along this parameter $\theta_{pl}, \Theta_n \in \Upsilon$. The modulus of these functions are bounded due to the choice of the variable $\alpha$, and inequality (32.2), the absolute values of the entries of the matrix

$$Y_n = -n^{-2} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - \hat{\Xi}_n] W_n(\alpha)[\Xi_n^{(j)} - \hat{\Xi}_n]\chi\{\Omega_n\}$$

will be smaller than the constant $c, 0<c<\alpha$. Therefore, these two graphs $y=\Im\theta_{pl}, -\alpha<\Im\theta_{pl}\le 0$ and $y=\{Y_n(\Im\theta_{pl})\}_{pl}$ will intersect in the square $-\alpha<\theta_{pl}\le 0, -\alpha<y\le 0$. Then there exists a solution for this component $\Im\theta_{pl}$ of the equation $K_{103}$ at any values of the other components when $\alpha>\beta_{\text{megic}}$. The same solution exists for the real part of the component $\theta_{pl}$. But this contradicts our assumption that this solution does not exist and we obtain that there exists a solution for all other entries $\theta_{pl}$ of matrices $\Theta_n^{(1)}, \Theta_n^{(2)}$ in the class of complex symmetrical matrices $\Upsilon$. Thus, the Lemma 33.3 is proved. □

In the same way, using equality

$$\bar{\Theta}_n^{(2)} = \mathbf{E} \sum_{j=1,\dots,n} [\Xi_n^{(j)} - A_n]\Im\mathbf{E}\,T_n(x+\mathrm{i}\alpha)[\Xi_n^{(j)} - A_n],$$

we prove:

**Lemma 33.4.** *Under conditions (33.3) and (33.4) there exists solution $\bar{\Theta}_n, I_n\alpha+\bar{\Theta}_n^{(2)}>0$ of the canonical equation $K_{104}$ in the class of complex symmetrical matrices $\Pi$.*

**Lemma 33.5.** *Under condition (33.1) and (33.2) the solution $\Theta_n$ of the canonical equation $K_{103}$ is unique in the class of matrices $\Upsilon$.*

*Proof.* We we'll argue the opposite. That is, let there exist two solutions $\Theta_n^{(u)}, \Theta_n^{(v)}$ of the canonical equation $K_{103}$.

Let $[\Theta_n^{(u)} - \Theta_n^{(v)}] = U_n\Lambda_n V_n$, where $U_n, V_n$ are Unitary matrices and $\Lambda_n = \{\lambda_i\delta_{ij}\}$ is the diagonal matrix of singular eigenvalues $\lambda_1 \cdots \ge \lambda_n$. Then we have from equation (32.2) that

$$\begin{aligned}
\lambda_1 &= \max_{\vec{x},\vec{y}:\vec{x}^{\,*}\vec{x}\le 1,\vec{y}^{\,*}\vec{y}\le 1} \vec{x}^{\,*}U_n\Lambda_nV_n\vec{y}\\
&= \max_{\vec{x},\vec{y}:\vec{x}^{\,*}\vec{x}\le 1,\vec{y}^{\,*}\vec{y}\le 1} \vec{x}^{\,*}n^{-2}\sum_{j=1,\dots,n}[\Xi_n^{(j)}-\hat{\Xi}_n]R_n(x+\mathrm{i}\alpha,\Theta_n^{(u)})U_n\Lambda_nV_n\\
&\times R_n(x+\mathrm{i}\alpha,\Theta_n^{(v)})[\Xi_n^{(j)}-\hat{\Xi}_n]\vec{y}\chi\{\Omega_n\}\\
&\le 4\alpha^{-2}n^{-2}\max_{\vec{x},\vec{y}:\vec{x}^{\,*}\vec{x}\le 1,\vec{y}^{\,*}\vec{y}\le 1}\sum_{j=1,\dots,n}\sqrt{\vec{x}^{\,*}[\Xi_n^{(j)}-\hat{\Xi}_n]^2\vec{x}}\sqrt{\vec{y}^{\,*}[\Xi_n^{(j)}-\hat{\Xi}_n]^2\vec{y}}\lambda_1\chi\{\Omega_n\}\\
&\le 4\alpha^{-2}n^{-2}\max_{\vec{x}:\vec{x}^{\,*}\vec{x}\le 1}\vec{x}^{\,*}\sum_{j=1,\dots,n}[\Xi_n^{(j)}-\hat{\Xi}_n]^2\vec{x}\chi\{\Omega_n\}\lambda_1\\
&\le 16\alpha^{-2}\beta_{\mathrm{magic}}\lambda_1\le c\lambda_1, c<1.
\end{aligned}$$

Therefore $\lambda_1=0$ and $\Theta_n^{(u)}=\Theta_n^{(v)}$. The resulting contradiction proves the Lemma 33.5. □

As in the same manner we have

**Lemma 33.6.** *Under the condition (33.3) and (33.4) the solution $\hat{\Theta}_n$ of the canonical equation $K_{104}$ is unique in the class of matrices $\Pi$.*

# 34 The estimator $G_{58}$ when the correlations of the entries of the matrices are known. Stochastic canonical equation $K_{105}$. Auxiliary system of canonical equations $K_{106}$. The formulations of the Theorems 34.1 and 34.2

We consider also the estimator $G_{58}=\Psi_n(x+\mathrm{i}\alpha)$ and $\Psi_n(x+\mathrm{i}\alpha)$ satisfies the system of canonical equations $K_{105}$

$$\Psi_n(x+\mathrm{i}\alpha)=n^{-2}\sum_{j=1,\dots,n}\mathbf{E}\{[\Xi_n^{(j)}-A_n][X_n][\Xi_n^{(j)}-A_n]\}_{X_n=M_n(x+\mathrm{i}\alpha)},\tag{34.1}$$

where

$$M_n(x+\mathrm{i}\alpha)=\left\{(x+\mathrm{i}\alpha)I_n+\Psi_n(x+\mathrm{i}\alpha)+n^{-1}\sum_{k=1,\dots,n}\Xi_n^{(k)}\right\}^{-1}.$$

and the auxiliary system of canonical equations $K_{106}$

$$\tilde{\Psi}_n(x+\mathrm{i}\alpha)=n^{-2}\mathbf{E}\sum_{j=1,\dots,n}[\Xi_n^{(j)}-A_n]\mathbf{E}\,M_n(x+\mathrm{i}\alpha)[\Xi_n^{(j)}-A_n],\tag{34.2}$$

where

$$M_n(x+\mathrm{i}\alpha)=\left\{(x+\mathrm{i}\alpha)I_n+\tilde{\Psi}_n(x+\mathrm{i}\alpha)+n^{-1}\sum_{k=1,\dots,n}\Xi_n^{(k)}\right\}^{-1}.$$

**Theorem 34.1.** *Let for any $n=1,2,..$ the matrices $\Xi_n^{(j)},j=1,...,n$ be independent, $\mathbf{E}\,\Xi_n^{(j)}=A_n,j=1,...,n$, let the second moments of the entries of the matrices $\Xi_n^{(j)},j=1,...,n$ be bounded, let condition (33.1) and (33.2) be fulfilled,*

$$\lim_{n\to\infty}n^{-7}\max_{i,j=1,..,n}\left\{\mathbf{E}\,\mathrm{Tr}\,([Y_n^{(i)}]^2[Y_n^{(j)}]^2)^2\right\}^{1/2}\left\{\mathbf{E}\,\mathrm{Tr}\,(\Xi_n^{(i)}+\Xi_n^{(j)})^4\right\}^{1/2}=0\tag{34.3}$$

*and*

$$\lim_{n\to\infty} n^{-6} \max_{k,j=1,\dots,n} \sum_{p,l=1,\dots,n} \operatorname{Tr} R^2_{pl,j} \operatorname{Tr} \mathbf{E}\, [\Xi_n^{(j)}]^4 = 0, \tag{34.4}$$

*where* $R_{pl,j} = \mathbf{E}\, \vec{\xi}_l^{\,(j)} \vec{\xi}_p^{\,(j)*}, \operatorname{Tr}(\mathbf{E}\,[Y^{(j)}]^2)^2 = \sum_{p,l=1,\dots,n} (\operatorname{Tr} R_{pl})^2,\ \vec{\xi}_l^{\,(j)}$ *are the vector column of the matrix* $Y^{(j)}$.

*Let the following conditions be fulfilled*

$$\max_{j,k=1,\dots,n} n^{-1} \lambda_j \{\mathbf{E}\, [Y_n^{(k)}]^2\} \le c,\ \lim_{n\to\infty} \max_{k=1,\dots,n} n^{-1} \operatorname{Tr}\, [A_{n\times n}]^4 \le c,\ \lim_{n\to\infty} \max_{k=1,\dots,n} n^{-3} \operatorname{Tr} \mathbf{E}\, [\Xi_n^{(k)}]^4 \le c, \tag{34.5}$$

*for any Hermitian matrices* $C_n^{(i)}, i = 1, 2$ *with bounded eigenvalues by some constant*

$$\lim_{n\to\infty} \max_{k=1,\dots,n} \max_{C_n^{(1)}:||C_n^{(1)}||\le 1} n^{-3} \mathbf{E}\, |\operatorname{Tr} Y_n^{(k)} C_n^{(1)}|^2 = 0, \tag{34.6}$$

$$\lim_{n\to\infty} \frac{1}{n^5} \max_{C_n^{(i)},k,j=1,\dots,n} \operatorname{Tr} \mathbf{E} \left\{ \mathbf{E}_{Y_n^{(j)}} Y_n^{(j)} C_n^{(1)} Y_n^{(k)} C_n^{(2)} Y_n^{(j)} \right\} \left\{ \mathbf{E}_{Y_n^{(j)}} Y_n^{(j)} C_n^{(1)} Y_n^{(k)} C_n^{(2)} Y_n^{(j)} \right\}^* = 0, \tag{34.7}$$

$$\lim_{n\to\infty} \frac{1}{n^3} \max_{k,j=1,\dots,n, j\ne k} \max_{C_n^{(1)}:||C_n^{(i)}||\le 1, i=1,2} |\mathbf{E} \operatorname{Tr} C_n^{(1)} Y_n^{(k)} C_n^{(2)} Y_n^{(j)} C_n^{(2)} Y_n^{(k)} C_n^{(2)} Y_n^{(j)}| = 0. \tag{34.8}$$

*Then for any* $x$

$$\sup_x \operatorname{l.i.m.}_{n\to\infty} [n^{-1} \operatorname{Tr} \{I_n(\mathrm{i}\alpha + x) + G_{57}(\mathrm{i}\alpha + x)\} - n^{-1} \operatorname{Tr} \{I_n(\mathrm{i}\alpha + x) + A_n\}] = 0.$$

**Theorem 34.2.** *Let for any* $n = 1, 2, ..$ *the matrices* $\Xi_n^{(j)}, j = 1, ..., n$ *be independent,* $\mathbf{E}\, \Xi_n^{(j)} = A_n, j = 1, ..., n$*, let the second moments of the entries of the matrices* $\Xi_n^{(j)}, j = 1, ..., n$ *be bounded, and let conditions (33.3), (33.4) and (34.3)–(34.8) be fulfilled. Then for any* $x$

$$\sup_x \operatorname{l.i.m.}_{n\to\infty} [n^{-1} \operatorname{Tr} \{I_n(\mathrm{i}\alpha + x) + G_{58}(\mathrm{i}\alpha + x)\} - n^{-1} \operatorname{Tr} \{I_n(\mathrm{i}\alpha + x) + A_n\}] = 0.$$

We will need the conditions (34.3)–(34.8) that may seem to be complex, although they are easily checked for some simple cases.

**Proof of the Theorem 34.1** As in the proof of Theorem 13.1 we follow several steps.

# 35 Converges of the difference $\hat{\Theta}_n - \Theta_n$ of the solutions of the equations $K_{103}$ and $K_{104}$ to zero

As in the proof of the consistency of the estimator $G_{55}$ we prove for singular eigenvalues $\lambda_k\{\sqrt{B_n B_n^*}\}, k = 1, ..., n$ of the matrices $B_n = T_n - R_n = R_n[\Theta_n - \hat{\Theta}_n]T_n$ the following statement

**Lemma 35.1.** *Under the conditions of Theorem 34.1*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1,\dots,n} \sup_x \mathbf{E}\, \lambda_k\{\sqrt{B_n B_n^*}\} = 0.$$

*Proof.* Remember that

$$\max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - \hat{X}_n]^2 \right\} \le 4 \max_{k=1,\dots,n} \lambda_k \left\{ \sum_{j=1,\dots,n} \frac{1}{n^2} [\Xi_n^{(j)} - A_n]^2 \right\}. \tag{35.1}$$

Since $B_n = U_n\Lambda_n V_n$, where $U_n, V_n$ are Unitary matrices and $\Lambda_n = \{\delta_{ki}\lambda_k\{\sqrt{B_nB_n^*}\}, k, i = 1, ..., n\}$. we get for any $\epsilon > 0$

$$|n^{-1}\mathrm{Tr}\,\{R_n - T_n\}| = n^{-1}|\mathrm{Tr}\,B_n| = n^{-1}|\mathrm{Tr}\,U_n\Lambda_n V_n| \leq n^{-1}\mathrm{Tr}\,\sqrt{B_nB_n^*}, \tag{35.2}$$

where since $\sum_{j=1,\ldots,n} n^{-2}R_n[\Xi_n^{(j)} - \hat{\Xi}_n]B_n[A_n - \hat{\Xi}_n]T_n = 0$, we have

$$\begin{aligned}
B_n \quad &= -\sum_{j=1,\ldots,n} n^{-2}R_n[\Xi_n^{(j)} - \hat{\Xi}_n]B_n[\Xi_n^{(j)} - \hat{\Xi}_n]T_n\chi\{\Omega_n\} + (T_n - R_n)\chi\{\bar{\Omega}_n\} \\
&-R_n\sum_{j=1,\ldots,n} n^{-2}\mathbf{E}_{\Xi^{(j)}}\{[\Xi_n^{(j)} - A_n][T_n - \mathbf{E}\,T_n][\Xi_n^{(j)} - A_n]\}T_n\chi\{\Omega_n\} \\
&+R_n\sum_{j=1,\ldots,n} n^{-2}\{[\Xi_n^{(j)} - A_n]T_n[\Xi_n^{(j)} - A_n] - \mathbf{E}_{\Xi^{(j)}}[\Xi_n^{(j)} - A_n]T_n[\Xi_n^{(j)} - A_n]\}T_n\chi\{\Omega_n\} \\
&-R_n[\hat{\Xi}_n - A_n]T_n[\hat{\Xi}_n - A_n]T_n\chi\{\Omega_n\},
\end{aligned} \tag{35.3}$$

where $\mathbf{E}_{Y_n^{(j)}}$ is the conditional expectation under fixed random matrices $Y_n^{(k)}, k \neq j$, and $Y_n^{(j)} = \Xi_n^{(j)} - A_n$.

Then since the matrix $\Lambda_n$ is the real matrix

$$n^{-1}\mathrm{Tr}\,\Lambda_n\chi\{\Omega_n\} = \Re\{K_1 + K_2 + K_3 + K_4\} + \epsilon_n,$$

where

$$\begin{aligned}
K_1 = \quad & n^{-1}\mathrm{Tr}\,\Phi_n\Lambda_n, \\
& \Phi_n = \sum_{j=1,\ldots,n} n^{-2}V_nT_n[\Xi_n^{(j)} - \hat{\Xi}_n]T_nV_n^*U_n^*R_n[\Xi_n^{(j)} - \hat{\Xi}_n]R_nU_n\chi\{\Omega_n\}, \\
& K_2 = n^{-1}\mathrm{Tr}\,U_n^*R_n\sum_{j=1,\ldots,n} n^{-2}\mathbf{E}_{\Xi^{(j)}}\{[\Xi_n^{(j)} - A_n][T_n - \mathbf{E}\,T_n][\Xi_n^{(j)} - A_n]\}T_nV_n^*\chi\{\Omega_n\} \\
& K_3 = n^{-1}\mathrm{Tr}\,U_n^*R_n\Bigg\{\sum_{j=1,\ldots,n} n^{-2}\{[\Xi_n^{(j)} - A_n]T_n[\Xi_n^{(j)} - A_n] \\
& -\mathbf{E}_{\Xi^{(j)}}[\Xi_n^{(j)} - A_n]T_n[\Xi_n^{(j)} - A_n]\}T_n\Bigg\}V_n^*\chi\{\Omega_n\}, \\
& K_4 = n^{-2}\mathrm{Tr}\,U_n^*R_n[\hat{\Xi}_n - A_n]T_n[\hat{\Xi}_n - A_n]T_nV_n^*\chi\{\Omega_n\}.
\end{aligned} \tag{35.4}$$

From this equality (35.4) we get using (35.1)–(35.3) and inequality $\mathrm{Tr}\,AL \leq \max\sqrt{\lambda(AA^*)}\mathrm{Tr}\,L$, where $L > 0$ is a real diagonal matrix with non negative diagonal entries and $A$ is the complex symmetrical matrix, that

$$\begin{aligned}
\sup_x \mathbf{E}\,|K_1| &\le \sup_x \mathbf{E} \max_{k=1,\dots,n} |\lambda_k\{\Phi_n\Phi_n^*\}|^{1/2} n^{-1}\mathrm{Tr}\,\Lambda_n \\
&= \sup_x \mathbf{E} \max_{\vec z,u:\vec z^{\,*}\vec z\le 1,\vec u^{\,*}\vec u\le 1} |\vec z^{\,*}\{\Phi_n\Phi_n^*\}\vec u| n^{-1}\mathrm{Tr}\,\Lambda_n \\
&\le \sup_x \mathbf{E} \max_{\vec z:\ \vec z^{\,*}\vec z\le 1} \sqrt{\vec z^{\,*}\sum_{j=1,\dots,n}\frac{1}{n^2} R_n[\Xi_n^{(j)}-\hat\Xi_n]R_nR_n^*[\Xi_n^{(j)}-\hat\Xi_n]R_n^*\vec z} \\
&\times\sqrt{\vec z^{\,*}\sum_{j=1,\dots,n}\frac{1}{n^2} T_n[\Xi_n^{(j)}-\hat\Xi_n]T_nT_n^*[\Xi_n^{(j)}-\hat\Xi_n]T_n^*\vec z}\ \ n^{-1}\mathrm{Tr}\,\Lambda_n\chi\{\Omega_n\} \\
&\le \frac{1}{\alpha^4}\sup_x \mathbf{E} \max_{k=1,\dots,n}\lambda_k\left\{\sum_{j=1,\dots,n}\frac{1}{n^2}[\Xi_n^{(j)}-\hat\Xi_n]^2\right\}\chi\{\Omega_n\}n^{-1}\mathrm{Tr}\,\Lambda_n \\
&\le \frac{2^6\beta^2_{\mathrm{magic}}}{\alpha^4} n^{-1}\sup_x \mathbf{E}\,\mathrm{Tr}\,\Lambda_n\chi\{\Omega_n\} \le cn^{-1}\sup_x \mathbf{E}\,\mathrm{Tr}\,\Lambda_n\chi\{\Omega_n\}, c<1. \qquad (35.5)
\end{aligned}$$

Similarly we get

$$K_2 = cn^{-1}\mathrm{Tr}\,\{D_nL_n\}, \max_x \mathbf{E}\,|K_2|^2 \le \max_x cn^{-1}\mathbf{E}\,\mathrm{Tr}\,D_nD_n^*n^{-1}\mathbf{E}\,\mathrm{Tr}\,L_nL_n^* \le cn^{-1}\max_x \mathbf{E}\,\mathrm{Tr}\,L_nL_n^*, \qquad (35.6)$$

where

$$D_n = T_nV_n^*U_n^*R_n, \quad L_n = \sum_{j=1,\dots,n} n^{-2}\mathbf{E}_{\,Y^{(j)}}Y_n^{(j)}[T_n-\mathbf{E}\,T_n]Y_n^{(j)}.$$

But for the expression $cn^{-1}\mathbf{E}\,\mathrm{Tr}\,L_nL_n^*$ we can use the theorem on the self-averaging of normalized traces of resolvents of random matrices. Therefore as in he calculation of $K_1$ we can use for the expression $cn^{-1}\mathbf{E}\,\mathrm{Tr}\,L_nL_n^*$ the theorem on the self-averaging of normalized traces of resolvents, namely for any Hermitian matrix $Q_n$ with nonnegative bounded eigenvalues using inequality $\mathrm{Tr}\,(\mathbf{E}\,B_n)^2 \le \mathbf{E}\,\mathrm{Tr}\,(B_n)^2$ for any Hermitian matrix

$$n^{-5}\mathbf{E}\,\mathrm{Tr}\,L_nL_n^* \le cn^{-7}\sum_{k=1,\dots,n}\mathbf{E}\,F_kF_k^* + cn^{-9}\sum_{k=1,\dots,n}\mathbf{E}\,N_kN_k^*,$$

where $\mathbf{E}_{\,k}$ is the conditional expectation under fixed matrices $\Xi_n^{(j)}, j=k+1,...,n,$

$$F_k = \sum_{j=1,\dots,n}[\mathbf{E}_{\,k-1}-\mathbf{E}_{\,k}]\mathbf{E}_{\,Y_n^{(j)}}Y_n^{(j)}\{T_n^{(k)}Y_n^{(k)}T_n^{(k)}\}Y_n^{(j)},$$

$$N_k = \sum_{j=1,\dots,n}[\mathbf{E}_{\,k-1}-\mathbf{E}_{\,k}]\mathbf{E}_{\,Y_n^{(j)}}Y_n^{(j)}\{T_n^{(k)}\Xi_n^{(k)}T_n^{(k)}\Xi_n^{(k)}T_n\}Y_n^{(j)}.$$

Hence

$$\begin{aligned}
n^{-5}\mathbf{E}\,\mathrm{Tr}\,L_nL_n^* &\le cn^{-5}\max_{k,j=1,\dots,n}\max_{Q_n:||Q_n||\le 1}\mathrm{Tr}\,\mathbf{E}\left\{\mathbf{E}_{Y_n^{(j)}}Y_n^{(j)}Q_nY_n^{(k)}Q_nY_n^{(j)}\right\}\left\{\mathbf{E}_{Y_n^{(j)}}Y_n^{(j)}Q_nY_n^{(k)}Q_nY_n^{(j)}\right\}^*\\
&+cn^{-6}\max_{k,j=1,\dots,n}\mathrm{Tr}\,\mathbf{E}\left\{\mathbf{E}\,\mathrm{Tr}\left[\left\{\mathbf{E}_{Y_n^{(j)}}\vec{\xi}_p^{\,(j)*}T_n^{(k)}\Xi_n^{(k)}T_n^{(k)}\Xi_n^{(k)}T_n\vec{\xi}_l^{(j)}\right\}_{p,l=1,\dots,n}\right]\right.\\
&\times\left[\left\{\mathbf{E}_{Y_n^{(j)}}\vec{\xi}_p^{\,(j)*}T_n^{(k)}\Xi_n^{(k)}T_n^{(k)}\Xi_n^{(k)}T_n\vec{\xi}_l^{(j)}\right\}_{p,l=1,\dots,n}\right]^*\\
&\le \epsilon_n+cn^{-6}\max_{k,j=1,\dots,n}\sum_{p,l=1,\dots,n}|\mathrm{Tr}\,R_{pl,j}T_n^{(k)}\Xi_n^{(k)}T_n^{(k)}\Xi_n^{(k)}T_n|^2\\
&\le \epsilon_n+cn^{-6}\max_{k,j=1,\dots,n}\sum_{p,l=1,\dots,n}\mathrm{Tr}\,R_{pl,j}^2\,\mathrm{Tr}\,\mathbf{E}\,[\Xi_n^{(j)}]^4\\
&\le \epsilon_n, \qquad (35.7)
\end{aligned}$$

Thus,

$$\sup_x \mathbf{E}\,[\Re K_2]^2\le cn^{-1}. \qquad (35.8)$$

Now we find inequality for $K_3$:

$$\begin{aligned}
\mathbf{E}\,[\Re K_3]^2\le\quad &\le n^{-5}\mathbf{E}\sum_{i,j=1,\dots,n}\mathrm{Tr}\left\{Y_n^{(i)}T_nY_n^{(i)}-\mathbf{E}_{Y^{(i)}}T_nY_n^{(i)}\right\}\\
&\times\left\{Y_n^{(j)}T_nY_n^{(j)}-\mathbf{E}_{Y^{(j)}}Y_n^{(j)}T_nY_n^{(j)}\right\}^*\\
&=M_n+P_n+S_n, \qquad (35.9)
\end{aligned}$$

where

$$\begin{aligned}
M_n\quad &=2n^{-6}\sum_{i,j=1,\dots,n}\mathbf{E}\,\mathrm{Tr}\left\{Y_n^{(i)}T_n^{(i,j)}Y_n^{(i)}-\mathbf{E}_{Y^{(i)}}Y_n^{(i)}T_n^{(i,j)}Y_n^{(i)}\right\}^*\\
&\times\left\{Y_n^{(j)}\Psi_n^{(i,j)}Y_n^{(j)}-\mathbf{E}_{Y^{(j)}}Y_n^{(j)}\Psi_n^{(i,j)}Y_n^{(j)}\right\}, \qquad (35.10)
\end{aligned}$$

$$P_n=n^{-5}\sum_{i=1,\dots,n}\mathbf{E}\,\mathrm{Tr}\left\{Y_n^{(i)}T_n^{(i,i)}Y_n^{(i)}-\mathbf{E}_{Y^{(i)}}Y_n^{(i)}T_n^{(i,i)}Y_n^{(i)}\right\}\left\{Y_n^{(i)}T_n^{(i,i)}Y_n^{(i)}-\mathbf{E}_{Y^{(i)}}Y_n^{(i)}T_n^{(i,i)}Y_n^{(i)}\right\}^*,$$

$$\begin{aligned}
S_n\quad &=n^{-7}\sum_{i,j=1,\dots,n}\mathbf{E}\,\mathrm{Tr}\left\{Y_n^{(i)}\Psi_n^{(i,j)}Y_n^{(i)}-\mathbf{E}_{Y^{(i)}}Y_n^{(i)}\Psi_n^{(i,j)}Y_n^{(i)}\right\}\\
&\times\left\{Y_n^{(j)}\Psi_n^{(i,j)}Y_n^{(j)}-\mathbf{E}_{Y^{(j)}}Y_n^{(j)}\Psi_n^{(i,j)}Y_n^{(j)}\right\}^*, \qquad (35.11)
\end{aligned}$$

where $\Psi_n^{(i,j)}=T_n^{(i,j)}(\Xi_n^{(i)}+\Xi_n^{(j)})T_n, T_n^{(i,j)}=[I_n(\alpha+\mathrm{i}x)+A_n+\tilde{\Theta}_n+\sum_{k\ne i,j,k=1,\dots,n}Y_n^{(k)}]^{-1}$.

Then

$$
\begin{aligned}
M_n \quad & \le cn^{-4} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, T_n Y_n^{(j)} Y_n^{(i)} T_n^{(i,j)} T_n^{(i,j)*} Y_n^{(i)} Y_n^{(j)} T_n^* \right\}^{1/2} \\
& \times \left\{ \mathbf{E}\,\mathrm{Tr}\, Y_n^{(i)} Y_n^{(j)} T_n^{(i,j)} [\Xi_n^{(j)} + \Xi_n^{(i)}]^2 T_n^{(i,j)*} Y_n^{(j)} Y_n^{(i)} \right\}^{1/2} \\
& \le cn^{-4} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, [Y_n^{(j)}]^2 [Y_n^{(i)}]^2 \right\}^{1/2} \\
& \times \left\{ \mathbf{E}\,\mathrm{Tr}\, [Y_n^{(j)}]^2 [Y_n^{(i)}]^2 T_n^{(i,j)} [\Xi_n^{(j)} + \Xi_n^{(i)}]^2 T_n^{(i,j)*} \right\}^{1/2} \\
& \le cn^{-4} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, [Y_n^{(j)}]^2 [Y_n^{(i)}]^2 \right\}^{1/2} \\
& \times \left\{ \mathbf{E}\,\mathrm{Tr}\, ([Y_n^{(i)}]^2 [Y_n^{(j)}]^2)^2 \right\}^{1/4} \left\{ \mathbf{E}\,\mathrm{Tr}\, (\Xi_n^{(i)} + \Xi_n^{(j)})^4 \right\}^{1/4} \\
& \le \frac{c\sqrt{n}}{n^4} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, ([Y_n^{(i)}]^2 [Y_n^{(j)}]^2)^2 \right\}^{1/4} \left\{ \mathbf{E}\,\mathrm{Tr}\, (\Xi_n^{(i)} + \Xi_n^{(j)})^4 \right\}^{1/4} \le \epsilon_n. \qquad (35.12)
\end{aligned}
$$

Analogously we prove

$$
\lim_{n\to\infty} P_n \le c \lim_{n\to\infty} n^{-4} \max_{i=1,..,n} \mathbf{E}\,\mathrm{Tr}\, [Y_n^{(i)}]^4 = 0.
$$

The next inequality more complicated. Using inequality (33.4)

$$
\begin{aligned}
S_n \quad & \le cn^{-7} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, Y_n^{(j)} Y_n^{(i)} T_n^{(i,j)} (\Xi_n^{(i)} + \Xi_n^{(j)}) T_n T_n^* (\Xi_n^{(i)} + \Xi_n^{(j)}) T_n^{(i,j)*} Y_n^{(i)} Y_n^{(j)} \right\}^{1/2} \\
& \times \left\{ \mathbf{E}\,\mathrm{Tr}\, Y_n^{(i)} Y_n^{(j)} T_n^{(i,j)} (\Xi_n^{(i)} + \Xi_n^{(j)}) T_n T_n^* (\Xi_n^{(i)} + \Xi_n^{(j)}) T_n^{(i,j)*} Y_n^{(j)} Y_n^{(i)} \right\}^{1/2} \\
& \le cn^{-7} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, Y_n^{(i)} [Y_n^{(j)}]^2 Y_n^{(i)} T_n^{(i,j)} (\Xi_n^{(i)} + \Xi_n^{(j)})^2 T_n^{(i,j)*} \right\}^{1/2} \\
& \times \left\{ \mathbf{E}\,\mathrm{Tr}\, Y_n^{(i)} [Y_n^{(j)}]^2 Y_n^{(i)} T_n^{(i,j)} (\Xi_n^{(i)} + \Xi_n^{(j)})^2 T_n^{(i,j)} \right\}^{1/2} \\
& \le cn^{-7} \max_{i,j=1,..,n} \left\{ \mathbf{E}\,\mathrm{Tr}\, ([Y_n^{(i)}]^2 [Y_n^{(j)}]^2)^2 \right\}^{1/2} \left\{ \mathbf{E}\,\mathrm{Tr}\, (\Xi_n^{(i)} + \Xi_n^{(j)})^4 \right\}^{1/2} \le \epsilon_n, \qquad (35.13)
\end{aligned}
$$

where $\lim_{n\to\infty} \epsilon_n = 0$.

Then using the conditions (35.3)–(35.5) we obtain

$$
\lim_{n\to\infty} \sup_x \mathbf{E}\, |[\Re K_3]^2| = 0. \qquad (35.14)
$$

Thus,

$$
\sup_x \mathbf{E}\, [\Re K_2]^2 \le \sup_x cn^{-1}, \mathbf{E}\, [\Re(K_3 + K_3^*)]^2 \le cn^{-1}, \sup_x \mathbf{E}\, [\Re K_4]^2 \le \frac{c}{n^2} \mathrm{Tr}\, [\hat{\Xi}_n - A_n]^2 \le cn^{-1}. \qquad (35.15)
$$

Therefore $\sup_x \mathbf{E}\, |K_2 + K_3 + K_4| \le \epsilon_n$ and $\lim_{n\to\infty} \epsilon_n = 0$.

Then using (35.1)–(35.15) we get

$$
\begin{aligned}
n^{-1} \sup_x \mathbf{E}\,\mathrm{Tr}\, \Lambda_n \chi\{\Omega_n\} \quad & = \sup_x \mathbf{E}\, [K_1 + K_2 + K_3 + K_4] \chi\{\Omega_n\} + \epsilon_n \\
& \le cn^{-1} \sup_x \mathbf{E}\,\mathrm{Tr}\,\mathrm{Tr}\, \Lambda_n \chi\{\Omega_n\} + \epsilon_n \\
& \le cn^{-1} \sup_x \mathbf{E}\,\mathrm{Tr}\,\mathrm{Tr}\, \Lambda_n \chi\{\Omega_n\} + \epsilon_n, c < 1. \qquad (35.16)
\end{aligned}
$$

and under the conditions of Theorem 34.1 we have using inequalities (35.1)– (35.16) that

$$\lim_{n\to\infty}\sup_x \mathbf{E}\,|n^{-1}\mathrm{Tr}\,\{R_n - T_n\}| = 0.$$

□

So, we have approximated the resolvent $R_n$ in which the MAGIC estimator $\Theta_n$ is stochastically dependent on the matrices $\Xi_n^{(k)}$ (and this did not allow us to apply the limit theorems) to the resolvent of the matrix $T_n$ in which the MAGIC estimator $\hat{\Theta}_n$ is non-random and does not depend on the matrices $\Xi_n^{(k)}$.

Now our final statement reads:

**Lemma 35.2.** *Under the conditions of Theorem 34.1 the following statement is valid*

$$\lim_{n\to\infty}\sup_x \mathbf{E}\,|n^{-1}\mathrm{Tr}\,\{I_n(\mathrm{i}\alpha+x) + \frac{1}{n}\sum_{k=1,\dots,n}\Xi_n^{(k)} + \hat{\Theta}_n(\mathrm{i}\alpha+x)\} - n^{-1}\mathrm{Tr}\,\{I_n(\mathrm{i}\alpha+x)+A_n\}| = 0.$$

*Proof.* We completely repeat the proof of the Theorem 19.2 in which the resolvent $[-I_n z + n^{-1}\sum_{k=1,\dots,n}\Xi_n^{(k)}]^{-1}$ is replaced by $[I_n(\mathrm{i}\alpha+x)+\hat{\Theta}_n(\mathrm{i}\alpha+x)+n^{-1}\sum_{k=1,\dots,n}\Xi_n^{(k)}]^{-1}$. And as a result we obtain

$$\frac{1}{n}\sup_x \mathbf{E}\,|\mathrm{Tr}\,[T_n - \mathbf{E}\,T_n]|^2 \le \frac{c}{\sqrt{n}} \tag{35.17}$$

and

$$\begin{aligned}\frac{1}{n}\mathbf{E}\,\mathrm{Tr}\,T_n(\mathrm{i}\alpha+x) \quad &= \Bigg\{ I_n(\mathrm{i}\alpha+x)+A_n+\hat{\Theta}_n(\mathrm{i}\alpha+x) \\ &\quad -\frac{1}{n^2}\sum_{k=1,\dots,n}\mathbf{E}\,[\Xi_n^{(k)}-A_n]\mathbf{E}\,T_n(\mathrm{i}\alpha+x)[\Xi_n^{(k)}-A_n]\Bigg\}^{-1} + \epsilon_n \\ &= \{I_n(\mathrm{i}\alpha+x)+A_n\}^{-1}+\epsilon_n,\end{aligned}$$

where $\lim_{n\to\infty}|\varepsilon_n| = 0$, □

So, we have proved Theorem 34.1.

The proof of the Theorem 34.2 is similar and much simpler.

Now we give one simple example when the condition (33.1) of the Theorem 34.1 is satisfied.

**Corollary 35.3.** *Let the matrices $\Xi^{(j)}$ be equal to $\Xi^{(j)} - A_n = X^{(j)} + X^{(j)*}$, where the matrices $X^{(j)}$ are independent with independent entries with zero expectations and equal variance $\sigma > 0$ and bounded absolute moments of the order $4+\delta$ then condition (33.1) is valid.*

The proof follows from the fact that in this case

$$\lambda_{max}\left\{\sum_{j=1,\dots,n}[X^{(j)}+X^{(j)*}]^2\right\} \le 4\lambda_{max}\left\{Y_{n\times n^2}Y^*_{n\times n^2}\right\} \le c\lambda_{max}\left\{Z^*_{n^2\times n^2}Z_{n^2\times n^2}\right\},$$

where $Y_{n\times n^2} = \left\{X^{(1)}, X^{(2)}, ..., X^{(n)}\right\}$ is the rectangular matrix, $Z^*_{n^2\times n^2} = \left\{Y^*_{n\times n^2}, Y^{(2)*}_{n\times n^2}, ..., Y^{(n)*}_{n\times n^2}\right\}$ and $Y_{n\times n^2}, Y^{(j)}_{n\times n^2}$ are independent matrices and $Y^{(j)}_{n\times n^2}$ have the same distribution as the matrix $Y_{n\times n^2}$. Now the matrix $Z_{n^2\times n^2}$ is square matrix and for it we can use many results for its maximum singular value. We give the simplest of them from the book[4], chapter 5

$$p\lim_{n\to\infty} n^{-2}\lambda_{max}\left\{Z_{n^2\times n^2}Z^*_{n^2\times n^2}\right\} < c.$$

**Remark 35.4.** *We have received our main statement, but at the same time the parameter $\alpha^2$ should be more than $\beta$. We have already seen how to rid of this parameter in the estimator $G_{55}$. In the next section we will show how to do this and remove this parameter $\alpha$ from the estimator $G_{57}$.*

We have found a consistent estimator $G_{57}$ of the normalized resolvent of the matrix $A_n$ of the MAGIC theory based on independent observations $\Xi_n^{(k)}$. This estimator $G_{57}$ is very different from the standard estimator in statistics $n^{-1}\mathrm{Tr}\,[I_n(\mathrm{i}\alpha + x) + n^{-1}\sum_{k=1,\ldots,n}\Xi_n^{(k)}]^{-1}$.

The proof of the theorem given below almost completely coincides with the proof of the Theorem 29.3 when we use the equalities (26.5) and (26.6).

# 36 MAGIC estimator $G_{57}$ of normalized spectral function $F_n(x)$ of the matrix $A_n$

**Theorem 36.1.** *Let the conditions of Theorem 34.1 be satisfied and let the normalized spectral functions $F_n(x)$ of the matrix $A_n$ weekly converge to the distribution function $F(x)$. Then for all points $u, v$ of continuity of the function $F(x)$*

$$\lim_{L\to\infty}\lim_{T\to\infty}\mathrm{l.i.m.}_{n\to\infty}\{[G_n(u,L,T) - G_n(v,L,T)] - [F(u) - F(v)]\} = 0,$$

*where*

$$G_n(u,L,T) - G_n(v,L,T) = \frac{1}{2\pi}\left\{\int_0^L \frac{e^{-\mathrm{i}vs} - e^{-\mathrm{i}us}}{\mathrm{i}s}K(s,T)\mathrm{d}s + \int_0^L \frac{e^{-ivs} - e^{\mathrm{i}us}}{\mathrm{i}s}K(-s,T)\mathrm{d}s\right\},$$

*with*

$$K(s,T) = \frac{1}{2\pi}\left\{\int_{-T}^{T} e^{-\mathrm{i}xs}\Phi_1(x,\alpha)\mathrm{d}x\right\}\mathrm{d}x e^{s\alpha}\mathrm{d}s,\ K(-s,T) = \frac{1}{2\pi}\left\{\int_{-T}^{T} e^{-\mathrm{i}xs}\Phi_2(x,\alpha)\mathrm{d}x\right\}\mathrm{d}x e^{s\alpha}\mathrm{d}s,\ s > 0,$$

*and*

$$\Phi_1(x,\alpha) = -\mathrm{i}[n^{-1}\mathrm{Tr}\,\{I_n(\mathrm{i}\alpha + x) + G_{57}(x)\}^{-1}, \Phi_2(x,\alpha) = -\mathrm{i}[n^{-1}\mathrm{Tr}\,\{I_n(\mathrm{i}\alpha + x) + G_{57}(-\mathrm{i}\alpha - x)\}^{-1}.$$

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