Series IV, exercise 1 Let $x_0, x_1, \dots, y_0, y_1, \dots \in (0, \infty)$ be such that

$$\lim_{n\to\infty} \frac{x_n}{x_0 + x_1 + \cdots + x_n} = \lim_{n\to\infty} \frac{y_n}{y_0 + y_1 + \cdots + y_n} = 0.$$

Let $z_n = \sum_{i=0}^{n} x_i y_{n-i}$ for $n \in \mathbb{N} \cup \{0\}$. Show that

$$\lim_{n\to\infty} \frac{z_n}{z_0 + z_1 + \cdots + z_n} = 0.$$

Solution Let

$$X_n := x_0 + x_1 + \cdots + x_n,$$

$$Y_n := y_0 + y_1 + \cdots + y_n$$

and

$$Z_n := z_0 + z_1 + \cdots + z_n.$$

We have

$$Z_n = \sum_{m=0}^{n} z_m = \sum_{m=0}^{n} \sum_{i=0}^{m} x_i y_{m-i} = x_0 \sum_{m=0}^{n} y_m + x_1 \sum_{m=1}^{n} y_{m-1} + \dots x_n \sum_{m=n}^{n} y_{m-n} = x_0 Y_n + x_1 Y_{n-1} + \dots x_n Y_0.$$

Similarly,

$$Z_n = y_0 X_n + y_2 X_{n-1} + \cdots + y_n X_0.$$

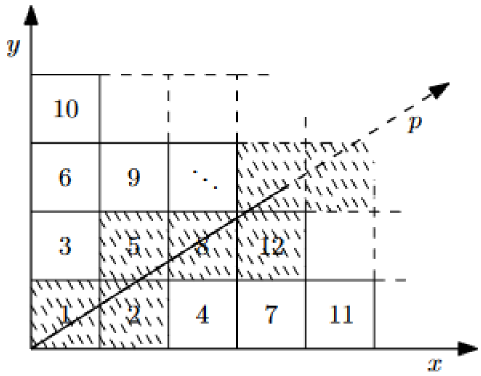
Let $\varepsilon > 0$. By the assumption, there is $N \in \mathbb{N}$ such that $\frac{x_n}{X_n} < \varepsilon$ and $\frac{y_n}{Y_n} < \varepsilon$ for $n \geq N$. Therefore, $x_n < \varepsilon X_n$, $y_n < \varepsilon Y_n$ for $n \geq N$. For $n \geq 2N$ we have

$$\begin{aligned}
\frac{z_n}{Z_n} &= \frac{x_0 y_n + x_1 y_{n-1} + \cdots + x_n y_0}{Z_n} \\
&\leq \frac{x_0 y_n + x_1 y_{n-1} + \cdots + x_{n-N} y_N}{Z_n} + \frac{x_N y_{n-N} + x_{N+1} y_{n-(N+1)} + \cdots + x_n y_0}{Z_n} \\
&= \frac{x_0 y_n + x_1 y_{n-1} + \cdots + x_{n-N} y_N}{x_0 Y_n + x_1 Y_{n-1} + \dots x_n Y_0} + \frac{x_N y_{n-N} + x_{N+1} y_{n-(N+1)} + \cdots + x_n y_0}{y_0 X_n + y_2 X_{n-1} + \cdots + y_n X_0} \\
&< \frac{x_0 y_n + x_1 y_{n-1} + \cdots + x_{n-N} y_N}{x_0 Y_n + x_1 Y_{n-1} + \dots x_{n-N} Y_N} + \frac{x_N y_{n-N} + x_{N+1} y_{n-(N+1)} + \cdots + x_n y_0}{X_N y_{n-N} + X_{N+1} y_{n-(N+1)} + \cdots + X_n y_0} \\
&< \frac{\varepsilon(x_0 Y_n + x_1 Y_{n-1} + \cdots + x_{n-N} Y_N)}{x_0 Y_n + x_1 Y_{n-1} + \dots x_{n-N} Y_N} + \frac{\varepsilon(X_N y_{n-N} + X_{N+1} y_{n-(N+1)} + \cdots + X_n y_0)}{X_N y_{n-N} + X_{N+1} y_{n-(N+1)} + \cdots + X_n y_0} = 2\varepsilon,
\end{aligned}$$

so $(\frac{z_n}{Z_n})$ is convergent to 0.

Series IV, exercise 2 Does there exist a countable set X and an uncountable family F of its subsets, such that for any $A, B \in F$, $A \neq B$ the set $A \cap B$ is finite?

Solution Put all natural numbers $\mathbb{N}$ in a coordinate system like in the picture.



For each ray p we put in the set A_p all numbers, which p intersects (intersects its square), and we place all such sets A_p into F. For each ray p we assign the angle α_p, which is between p and x. Since all α_p form the interval $(0, \pi_2)$, the set of rays p is uncountable. Furthermore, for different rays p_1 and p_2, the intersection $A_{p_1} \cap A_{p_2}$ is finite, because there exists distance $d(p_1, p_2)$ from the origin of the coordinate system where rays are far enough, so they will not pass through the same square any more. The problem is solved.

Series IV, exercise 3 Let $n \in \mathbb{N} \cup \{0\}$. Calculate

$$\sum_{i=0}^{n} 2^{n-i} \binom{n+i}{i}.$$

Solution Let $S(n) = \sum_{i=0}^{n} 2^{n-i} \binom{n+i}{i}$. Then

$$\begin{aligned} S(n+1) &= \sum_{i=0}^{n+1} 2^{n+1-i} \binom{n+1+i}{i} = \sum_{i=0}^{n+1} 2^{n+1-i} \left(\binom{n+i}{i} + \binom{n+i}{i-1} \right) \\ &= \sum_{i=0}^{n+1} 2^{n+1-i} \binom{n+i}{i} + \sum_{i=0}^{n} 2^{n-i} \binom{n+i+1}{i} \\ &= 2S(n) + \binom{2n+1}{n+1} + \frac{1}{2} \left(S(n+1) - \binom{2n+2}{n+1} \right) \\ &= 2S(n) + \frac{1}{2} S(n+1). \end{aligned}$$

Therefore $S(n+1) = 4S(n)$, and since $S(0) = 1$, by induction we obtain $S(n) = 4^n$.