G3-style Sequent Calculi and CIP for Logics with RDD

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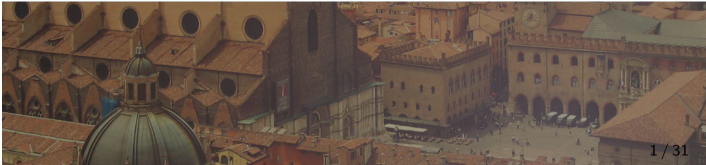
based on j.w.w. Norbert Gratzl (MCMP) & Edi Pavlović (Bayreuth)

ExtenDD
4 December 2024

DAR-Unibo



Foreword

- Thanks to Andrzej, Iaroslav, and Nils (and Michał)!

- Feel free to interrupt me at any time.

Goal

- The situation:
  - [2] Introduces RDD: DDS based on $\imath$ and λ that don't need a separate denotation clause for $\imath$-terms;
  - [1] gives G1-style calculus for classical FO-logic with RDD.
- Today:
  - We introduce a G3-calculus for it where all structural rules are (hp)-admissible;
  - We show the calculus has constructive CIP;
  - We extend the approach to positive and negative free logics;
  - we extend the approach to intuitionistic logic (with $\mathcal{E}$).

Plan

1. Classical FO-logic with RDD: FO^λ
2. Sequent calculus $\mathsf{G3c}^\lambda$
3. Classical free logics
4. Intuitionistic logic (with $\mathcal{E}$)

Plan

1 Classical FO-logic with RDD: FO^{λ}

2 Sequent calculus $\mathrm{G3c}^{\lambda}$

3 Classical free logics

4 Intuitionistic logic (with $\mathcal{E}$)

Motivation

- The formula $\neg\phi(\imath x.\psi))$ is ambiguous between inner and outer reading of negation.
- Russell's analysis:

$$\phi(\imath x.\psi) \equiv \exists x(\psi \wedge \forall y(\psi(y/x) \supset y = x) \wedge \phi)$$

gives scope to DDs by eliminating them.

- λ-abstraction gives scope to DDs without eliminating them:

$$\langle \lambda y.\phi(y/x)\rangle(\imath x.\neg\psi) \qquad \langle \lambda y.\neg\phi(y/x)\rangle(\imath x.\psi)$$

- [1, 2]: a semantics for λ-formulas that avoids a separate denotation clause for DDs, this simplifies proof systems!

Language

- Terms:

$$t ::= x \mid \imath x.\phi$$

- Formulas:

$$\phi ::= R(\vec{x}) \mid x = y \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid \forall x \phi \mid \exists x \phi \mid \langle \lambda x.\phi \rangle (t)$$

  where $R \in Rel^n$; $\vec{x}$ is an n-ary vector of variables; $x, y \in Var$; and t is a term.

- Observe that variables are the only terms occurring in atomic formulas.
- Formulas and substitutions are identified up to renaming of bound variables.[1]

[1] $\phi(y/x)$ always defined.

Semantics

- Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$, where $\mathcal{D} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}...\}$ is a set of objects and $\mathcal{I}$ an interpretation of predicates over them.
- Variables are directly mapped to objects.[2]
- Truth for λ-formulas:

$$\mathcal{M} \models \langle \lambda x.\phi \rangle(\mathbf{a}) \quad \textit{iff} \quad \mathcal{M} \models \phi(\mathbf{a}/x)$$

$$\mathcal{M} \models \langle \lambda x.\phi \rangle(\imath y.\psi) \;\; \textit{iff} \;\; \textit{some } \mathbf{a} \in \mathcal{D} \textit{ is s.t. } \mathcal{M} \models \phi(\mathbf{a}/x) \textit{ and it is the only object in } \mathcal{D} \textit{ s.t.:} \mathcal{M} \models \psi(\mathbf{a}/y)$$

[2] $\mathcal{M} \models \forall x \phi \quad \text{iff} \quad \text{for all } \mathbf{a} \in \mathcal{D}(\mathcal{M} \models \phi(\mathbf{a}/x).)$

Axiomatic calculus

We extend a standard axiomatisation of classical FO-logic with:

$$\langle \lambda x.\phi \rangle(y) \supset\subset \phi(y/x) \qquad (\beta\text{-red})$$

$$\langle \lambda x.\phi \rangle(\imath y.\psi) \supset\subset \exists y(\phi(y/x) \wedge \psi \wedge \forall z(\psi(z/y) \supset z = y)) \quad (\imath\text{-red})$$

Plan

The calculus G3c

Initial sequents: $P, \Gamma \Rightarrow \Delta, P$

Logical rules:

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot \qquad \frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}\ L\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}\ R\wedge \qquad \frac{\phi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \psi, \Gamma \Rightarrow \Delta}{\phi \supset \psi, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \supset \psi}\ R\supset \qquad \frac{\phi(z/x), \forall x\phi, \Gamma \Rightarrow \Delta}{\forall x\phi, \Gamma \Rightarrow \Delta}\ L\forall$$

$$\frac{\Gamma \Rightarrow \Delta, \phi(y/x)}{\Gamma \Rightarrow \Delta, \forall x\phi}\ R\forall,\ y! \qquad \frac{\phi(y/x), \Gamma \Rightarrow \Delta}{\exists x\phi, \Gamma \Rightarrow \Delta}\ L\exists,\ y! \qquad \frac{\Gamma \Rightarrow \Delta, \exists x\phi, \phi(z/x)}{\Gamma \Rightarrow \Delta, \exists x\phi}\ R\exists$$

Rules for identity:

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref \qquad \frac{P(z/w), x = z, P(x/w), \Gamma \Rightarrow \Delta}{x = z, P(x/w), \Gamma \Rightarrow \Delta}\ Repl$$

rule for λ-formulas: variables

- From the semantic clause

$$\mathcal{M} \models \langle \lambda x.\phi \rangle(\mathbf{a}) \quad \textit{iff} \quad \mathcal{M} \models \phi(\mathbf{a}/x)$$

- we get the following rules:

$$\frac{\phi(z/x), \Gamma \Rightarrow \Delta}{\langle \lambda x.\phi \rangle(z), \Gamma \Rightarrow \Delta}\ L\lambda_x \qquad \frac{\Gamma \Rightarrow \Delta, \phi(z/x)}{\Gamma \Rightarrow \Delta, \langle \lambda x.\phi \rangle(z)}\ R\lambda_x$$

rule for λ-formulas: ι-terms

- From the semantic clause

$$\mathcal{M} \models \langle \lambda x.\phi \rangle (\iota y.\psi) \text{ iff some } \mathbf{a} \in \mathcal{D} \text{ is s.t. } \mathcal{M} \models \phi(\mathbf{a}/x) \text{ and } \mathcal{M} \models \psi(\mathbf{a}/y) \text{ and for all } \mathbf{b} \in \mathcal{D},\ \mathcal{M} \models \psi(\mathbf{b}/y) \text{ implies } \mathbf{b} = \mathbf{a}$$

- we get the following rules (the left is a system of rules, cf. [3]):

$$\frac{\Gamma \Rightarrow \Delta, \langle \lambda x.\phi \rangle (\iota z.\psi), \phi(w/x) \quad \Gamma \Rightarrow \Delta, \langle \lambda x.\phi \rangle (\iota z.\psi), \psi(w/z) \quad \psi(y/z), \Gamma \Rightarrow \Delta, \langle \lambda x.\phi \rangle (\iota z.\psi), y = w}{\Gamma \Rightarrow \Delta, \langle \lambda x.\phi \rangle (\iota z.\psi)} \; R\lambda_{\iota}, y!$$

$$\frac{\Pi \Rightarrow \Sigma, \psi(w/z) \quad w = y, \Pi \Rightarrow \Sigma}{\Pi \Rightarrow \Sigma} \; L\lambda_{\iota}^{2}$$

$$\vdots$$
$$\mathcal{D}$$
$$\vdots$$

$$\frac{\phi(y/x), \psi(y/z), \Gamma \Rightarrow \Delta}{\langle \lambda x.\phi \rangle (\iota z.\psi), \Gamma \Rightarrow \Delta} \; L\lambda_{\iota}^{1}, y \text{ eigenvariable}$$

rule for λ-formulas: $\imath$-terms

- From the semantic clause

$$\mathcal{M} \models \langle \lambda x.\phi \rangle (\imath y.\psi) \text{ iff some } \mathbf{a} \in \mathcal{D} \text{ is s.t. } \mathcal{M} \models \phi(\mathbf{a}/x) \text{ and } \mathcal{M} \models \psi(\mathbf{a}/y) \text{ and for all } \mathbf{b} \in \mathcal{D},\ \mathcal{M} \models \psi(\mathbf{b}/y) \text{ implies } \mathbf{b} = \mathbf{a}$$

- we get the following rules (the left is a system of rules, cf. [3]):

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$$\cfrac{\cfrac{\Pi \Rightarrow \Sigma, \psi(w/z) \quad w = y, \Pi \Rightarrow \Sigma}{\Pi \Rightarrow \Sigma}\ L\lambda_\iota^2}{\vdots \; \mathcal{D} \; \vdots}$$

$$\frac{\phi(y/x), \psi(y/z), \Gamma \Rightarrow \Delta}{\langle \lambda x.\phi \rangle (\imath z.\psi), \Gamma \Rightarrow \Delta}\ L\lambda_\iota^1, y \text{ eigenvariable}$$

hsp-admissibility

Let $\mathcal{S}, \mathcal{S}_1, \ldots \mathcal{S}_m$ be sequents, and let

$$\frac{\mathcal{S}_1 \quad \cdots \quad \mathcal{S}_m}{\mathcal{S}} \; \mathit{Rule}$$

be a rule. We say that rule *Rule* is:

- *Admissible* if $\vdash \mathcal{S}_1$ and ... and $\vdash \mathcal{S}_n$ imply $\vdash \mathcal{S}$;
- *System-preserving admissible* if, moreover, the transformation used to show its admissibility doesn't impair the order of application of related instances of rules $L\lambda^1_\gamma$ and $L\lambda^2_\gamma$;
- Height- and system-preserving admissible (*hsp-admissible*, for short.) if $\vdash^n \mathcal{S}_1$ and ... and $\vdash^n \mathcal{S}_m$ imply $\vdash^n \mathcal{S}$ and the transformation is sytem-preserving;
- Height- and system-preserving invertible (*hsp-invertible*) if $\vdash^n \mathcal{S}$ implies $\vdash^n \mathcal{S}_1$ and ... and $\vdash^n \mathcal{S}_m$ and the transformation is system-preserving.

Structural rules and completeness

- The structural rules of substitution, weakening and contraction are hsp-admissible.
- all rules of $G3c^{\lambda}$ are hsp-invertible.
- Cut is syntactically admissible.
- $G3c^{\lambda} \vdash \Rightarrow \phi$ iff $FO^{\lambda} \vdash \phi$.

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- The structural rules of substitution, weakening and contraction are hsp-admissible.
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Split-interpolant

1. $L(\phi)$ is the set-theoretic union of all variables occurring free in ϕ and of all non-logical relational symbols occurring in ϕ, and the same for $\mathcal{L}(\Gamma)$;
2. A partition of a sequent $\Gamma \Rightarrow \Delta$ is an expression $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ such that $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$;
3. A split-interpolant of a partition $\Gamma_1 \ ; \ \Gamma_2 \Rightarrow \Delta_1 \ ; \ \Delta_2$ is a formula ξ such that:
   1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \xi$
   2. $\vdash \xi, \Gamma_2 \Rightarrow \Delta_2$
   3. $\mathcal{L}(\xi) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$.

Maehara's lemma and CIP

Lemma (Maehara). *Every partition* $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ *of a* $\mathsf{G3c}^{\lambda}$*-derivable sequent* $\Gamma \Rightarrow \Delta$ *has a split-interpolant.*

Proof. An algorithm calculating split-interpolants of the partitions of the conclusion from the split-interpolants of appropriate partitions of the premisses. □

Theorem (CIP). *If* $\mathsf{FO}^{\lambda} \vdash \phi \supset \psi$*, then there is* ξ *such that* $\mathsf{FO}^{\lambda} \vdash \phi \supset \xi$, $\mathsf{FO}^{\lambda} \vdash \xi \supset \psi$*, and* $\mathcal{L}(\xi) \subseteq \mathcal{L}(\phi) \cap \mathcal{L}(\psi)$.

Proof. Apply Maehara's lemma to $\phi \,;\, \emptyset \Rightarrow \emptyset \,;\, \psi$. □

Plan

FO-free logics

- Free logic is FO-logic where terms have no existential presupposition.[3]
- Its language is that of FO-logic plus the existence predicate $\mathcal{E}$.
- A model is triple $\langle \mathcal{D}, \mathcal{Q}, \mathcal{I} \rangle$ where $\mathcal{Q}$ — the quantifiers' range — is a subset of $\mathcal{D}$.
- In positive free logic (PF) predicates range over $\mathcal{D}$.
- in negative free logic (NF) predicates range over $\mathcal{Q}$.

[3] $\forall x \phi$ might be true and $\phi(y/x)$ false.

$\phi ::= \mathcal{E}x|\mathcal{R}(\vec{x}) \mid x = y \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid \forall x \phi \mid \exists x \phi \mid \langle \lambda x.\phi \rangle (t)$

$\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots$ *are members of* $\mathcal{D}$.

- In PF we have:

$$\begin{array}{lcl} \mathcal{M} \models \mathcal{E}\mathbf{a} & \text{iff} & \mathbf{a} \in \mathcal{Q} \\ \mathcal{M} \models \forall x \phi & \text{iff} & \text{for all } \mathbf{a} \in \mathcal{Q},\ \phi(\mathbf{a}/x) \\ \mathcal{M} \models \exists x \phi & \text{iff} & \text{for some } \mathbf{a} \in \mathcal{Q},\ \phi(\mathbf{a}/x) \end{array}$$

- In NF we change the atomic clauses:

$$\begin{array}{lcl} \mathcal{M} \models P(\vec{\mathbf{a}}) & \text{iff} & \vec{\mathbf{a}} \in \mathcal{I}(P) \text{ and } \vec{\mathbf{a}} \in \mathcal{Q} \\ \mathcal{M} \models \mathbf{a} = \mathbf{b} & \text{iff} & \mathbf{a} = \mathbf{b} \text{ and } \mathbf{a} \in \mathcal{Q} \end{array}$$

Language and Truth

$$\phi ::= \mathcal{E}x|\mathcal{R}(\vec{x}) \mid x = y \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid \forall x \phi \mid \exists x \phi \mid \langle \lambda x.\phi \rangle (t)$$

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- In PF we have:

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Adding definite descriptions

- XF_R extends XF with Russellian λ-formulas:

$$\mathcal{M} \models \langle \lambda x.\phi \rangle(\mathbf{a}) \quad \text{iff} \quad \mathcal{M} \models \phi(\mathbf{a}/x) \text{ and } \mathbf{a} \in \mathcal{Q}$$

$$\mathcal{M} \models \langle \lambda x.\phi \rangle(\imath y.\psi) \text{ iff some } \mathbf{a} \in \mathcal{Q} \text{ is s.t. } \mathcal{M} \models \phi(\mathbf{a}/x) \text{ and it is the only object in } \mathcal{Q} \text{ s.t.:} \mathcal{M} \models \psi(\mathbf{a}/y)$$

- XF_M extends XF with Meinongian λ-formulas:

$$\mathcal{M} \models \langle \lambda x.\phi \rangle(\mathbf{a}) \quad \text{iff} \quad \mathcal{M} \models \phi(\mathbf{a}/x)$$

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Axiomatisation of XF_R

- For PF_R we extend an axiomatisation of PF with:[4]

$(\beta/\mathcal{E}\text{-red}) \quad \langle \lambda x.\phi \rangle (y) \supset\subset \phi(y/x) \wedge \mathcal{E}y$

$(\imath/\exists\text{-red}) \; \langle \lambda x.\phi \rangle (\imath y.\psi) \supset\subset \exists y(\phi(y/x) \wedge \psi \wedge \forall z(\psi(z/y) \supset z = y))$

- For NF_R we extend an axiomatisation of PF_R (minus axiom $x = x$) with:

$(\text{atom-}\mathcal{E}) \quad P[x] \supset \mathcal{E}x$

$(\text{Ref}_=/\mathcal{E}) \quad \mathcal{E}x \supset x = x$

[4] $UI^{\mathcal{E}} := \mathcal{E}y \wedge \forall x \phi \supset \phi(y/x)$

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[4] $UI^{\mathcal{E}} := \mathcal{E}y \wedge \forall x \phi \supset \phi(y/x)$

Axiomatisation of XF_M

An axiomatisation of XF_M extends an axiomatisation of XF with:

- Axioms for the classical quantifiers Σ and Π
- axiom (β-red): $\langle \lambda x \phi \rangle (z) \supset\subset \phi(z/x)$
- axiom (ι/Σ-red):[5]

$$\langle \lambda x.\phi \rangle (\iota y.\psi) \supset\subset \Sigma y(\phi(y/x) \wedge \psi \wedge \Pi z(\psi(z/y) \supset z = y))$$

[5] Lambert's Law: $\forall y(\langle \lambda z.z = y \rangle (\iota x.\psi) \supset\subset \forall x(\psi \supset\subset x = y))$ is a thm. of XF_R BUT NOT of XF_M.

Sequent calculi G3xf$_R^\lambda$

- G3pf$_R^\lambda$ is obtained by extending G3p with (Repl) and:

$$\frac{\phi(z/x), \mathcal{E}z, \forall x\phi, \Gamma \Rightarrow \Delta}{\mathcal{E}z, \forall x\phi, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\mathcal{E}y, \Gamma \Rightarrow \Delta, \phi(y/x)}{\Gamma \Rightarrow \Delta, \forall x\phi}\ R\forall, y!$$

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$$\frac{\mathcal{E}z, \phi(z/x), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(z), \Gamma \Rightarrow \Delta}\ L\lambda_x \qquad \frac{\mathcal{E}z, \Gamma \Rightarrow \Delta, \phi(z/x)}{\mathcal{E}z, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(z)}\ R\lambda_x$$

$$\frac{\mathcal{E}y, \phi(y/x), \psi(y/z), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(\imath z.\psi), \Gamma \Rightarrow \Delta}\ L\lambda^1_\imath, y! \qquad \frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \psi(w/z) \quad w = y, \mathcal{E}w, \Gamma \Rightarrow \Delta}{\mathcal{E}w, \Gamma \Rightarrow \Delta}\ L\lambda^2_\imath$$

$$\frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \phi(w/x) \quad \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \psi(w/z) \quad \psi(y/z), \mathcal{E}y, \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), y = w}{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi)}\ R\lambda_\imath, y!$$

- G3nf$_R^\lambda$ extends G3pf$_R^\lambda$ with:

$$\frac{\mathcal{E}x, P[x], \Gamma \Rightarrow \Delta}{P[x], \Gamma \Rightarrow \Delta}\ At_{\mathcal{E}} \qquad \frac{\mathcal{E}x, x = x, \Gamma \Rightarrow \Delta}{\mathcal{E}x, \Gamma \Rightarrow \Delta}\ Ref_{\mathcal{E}}$$

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$$\frac{\mathcal{E}y, \phi(y/x), \psi(y/z), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(\imath z.\psi), \Gamma \Rightarrow \Delta}\ L\lambda^1_\imath, y! \qquad \frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \psi(w/z) \quad w = y, \mathcal{E}w, \Gamma \Rightarrow \Delta}{\mathcal{E}w, \Gamma \Rightarrow \Delta}\ L\lambda^2_\imath$$

$$\frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \phi(w/x) \quad \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \psi(w/z) \quad \psi(y/z), \mathcal{E}y, \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), y = w}{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi)}\ R\lambda_\imath, y!$$

- G3nf$^{\lambda}_{R}$ extends G3pf$^{\lambda}_{R}$ with:

$$\frac{\mathcal{E}x, P[x], \Gamma \Rightarrow \Delta}{P[x], \Gamma \Rightarrow \Delta}\ At_{\mathcal{E}} \qquad \frac{\mathcal{E}x, x = x, \Gamma \Rightarrow \Delta}{\mathcal{E}x, \Gamma \Rightarrow \Delta}\ Ref_{\mathcal{E}}$$

Sequent calculi G3xf$^{\lambda}_{R}$

- G3pf$^{\lambda}_{R}$ is obtained by extending G3p with (Repl) and:

$$\frac{\phi(z/x), \mathcal{E}z, \forall x\phi, \Gamma \Rightarrow \Delta}{\mathcal{E}z, \forall x\phi, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\mathcal{E}y, \Gamma \Rightarrow \Delta, \phi(y/x)}{\Gamma \Rightarrow \Delta, \forall x\phi}\ R\forall, y!$$

$$\frac{\mathcal{E}y, \phi(y/x), \Gamma \Rightarrow \Delta}{\exists x\phi, \Gamma \Rightarrow \Delta}\ L\exists, y! \qquad \frac{\mathcal{E}z, \Gamma \Rightarrow \Delta, \exists x\phi, \phi(z/x)}{\mathcal{E}z, \Gamma \Rightarrow \Delta, \exists x\phi}\ R\exists$$

$$\frac{\mathcal{E}z, \phi(z/x), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(z), \Gamma \Rightarrow \Delta}\ L\lambda_x \qquad \frac{\mathcal{E}z, \Gamma \Rightarrow \Delta, \phi(z/x)}{\mathcal{E}z, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(z)}\ R\lambda_x$$

$$\frac{\mathcal{E}y, \phi(y/x), \psi(y/z), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(\imath z.\psi), \Gamma \Rightarrow \Delta}\ L\lambda^1_\imath, y! \qquad \frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \psi(w/z) \quad w = y, \mathcal{E}w, \Gamma \Rightarrow \Delta}{\mathcal{E}w, \Gamma \Rightarrow \Delta}\ L\lambda^2_\imath$$

$$\frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \phi(w/x) \quad \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \psi(w/z) \quad \psi(y/z), \mathcal{E}y, \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), y = w}{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi)}\ R\lambda_\imath, y!$$

- G3nf$^{\lambda}_{R}$ extends G3pf$^{\lambda}_{R}$ with:

$$\frac{\mathcal{E}x, P[x], \Gamma \Rightarrow \Delta}{P[x], \Gamma \Rightarrow \Delta}\ At_{\mathcal{E}} \qquad \frac{\mathcal{E}x, x = x, \Gamma \Rightarrow \Delta}{\mathcal{E}x, \Gamma \Rightarrow \Delta}\ Ref_{\mathcal{E}}$$

Sequent calculi $G3xf^{\lambda}_{R}$

- $G3pf^{\lambda}_{R}$ is obtained by extending G3p with (Repl) and:

$$\frac{\phi(z/x), \mathcal{E}z, \forall x\phi, \Gamma \Rightarrow \Delta}{\mathcal{E}z, \forall x\phi, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\mathcal{E}y, \Gamma \Rightarrow \Delta, \phi(y/x)}{\Gamma \Rightarrow \Delta, \forall x\phi}\ R\forall, y!$$

$$\frac{\mathcal{E}y, \phi(y/x), \Gamma \Rightarrow \Delta}{\exists x\phi, \Gamma \Rightarrow \Delta}\ L\exists, y! \qquad \frac{\mathcal{E}z, \Gamma \Rightarrow \Delta, \exists x\phi, \phi(z/x)}{\mathcal{E}z, \Gamma \Rightarrow \Delta, \exists x\phi}\ R\exists$$

$$\frac{\mathcal{E}z, \phi(z/x), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(z), \Gamma \Rightarrow \Delta}\ L\lambda_x \qquad \frac{\mathcal{E}z, \Gamma \Rightarrow \Delta, \phi(z/x)}{\mathcal{E}z, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(z)}\ R\lambda_x$$

$$\frac{\mathcal{E}y, \phi(y/x), \psi(y/z), \Gamma \Rightarrow \Delta}{\langle\lambda x.\phi\rangle(\imath z.\psi), \Gamma \Rightarrow \Delta}\ L\lambda^1_\imath, y! \qquad \frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \psi(w/z) \quad w = y, \mathcal{E}w, \Gamma \Rightarrow \Delta}{\mathcal{E}w, \Gamma \Rightarrow \Delta}\ L\lambda^2_\imath$$

$$\frac{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \phi(w/x) \quad \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), \psi(w/z) \quad \psi(y/z), \mathcal{E}y, \mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi), y = w}{\mathcal{E}w, \Gamma \Rightarrow \Delta, \langle\lambda x.\phi\rangle(\imath z.\psi)}\ R\lambda_\imath, y!$$

- $G3nf^{\lambda}_{R}$ extends $G3pf^{\lambda}_{R}$ with:

$$\frac{\mathcal{E}x, P[x], \Gamma \Rightarrow \Delta}{P[x], \Gamma \Rightarrow \Delta}\ At_{\mathcal{E}} \qquad \frac{\mathcal{E}x, x = x, \Gamma \Rightarrow \Delta}{\mathcal{E}x, \Gamma \Rightarrow \Delta}\ Ref_{\mathcal{E}}$$

Sequent calculi $\mathsf{G3xf}^{\lambda}_{D}$

- The calculus $\mathsf{G3xf}^{\lambda}_{M}$ is obtained from $\mathsf{G3xf}^{\lambda}_{R}$ by removing active and principal $\mathcal{E}$-atoms from the rules for λ-formulas.[6]

- We use $\mathsf{G3xf}^{\lambda}_{X}$ to denote a generic calculus for a free logic with DDs.

[6] We have the rulef for $\forall/\exists$ of G3xf and the rules for λ-formulas of $\mathsf{G3c}^{\lambda}$.

main results

- All rules of $\mathsf{G3xf}^{\lambda}_{X}$ are hsp-invertible;
- Weakening and contraction are hsp-admissible in $\mathsf{G3xf}^{\lambda}_{X}$;
- Cut is syntactically admissible in $\mathsf{G3xf}^{\lambda}_{X}$;
- The calculus $\mathsf{G3xf}^{\lambda}_{X}$ is sound and complete;
- Maehara's lemma holds in $\mathsf{G3xf}^{\lambda}_{X}$.[7]

[7] Se that CIP holds in $\mathsf{XF}_{R/M}$.

Plan

Intuitionistic sequent calculus G3i$^\lambda$

- A single succedent intuitionistic sequent has shape:

$$\Gamma \Rightarrow \phi$$

- The calculus G3i$^\lambda$ is obtained from G3c$^\lambda$ by considering single conclusion version of its rules with the following modifications:

$$\frac{\Gamma \Rightarrow \phi_i}{\Gamma \Rightarrow \phi_1 \vee \phi_2}\ R\vee, i\in\{1,2\} \qquad \frac{\phi \supset \psi, \Gamma \Rightarrow \phi \quad \psi, \Gamma \Rightarrow \xi}{\phi \supset \psi, \Gamma \Rightarrow \xi}\ L\supset$$

$$\frac{\Gamma \Rightarrow \phi(z/x)}{\Gamma \Rightarrow \exists x\phi}\ R\exists \qquad \frac{\Gamma \Rightarrow \psi(w/z) \quad w = y, \Gamma \Rightarrow \xi}{\Gamma \Rightarrow \xi}\ L\lambda^2_\imath$$

$$\frac{\Gamma \Rightarrow \phi(w/x) \quad \Gamma \Rightarrow \psi(w/z) \quad \psi(y/z), \Gamma \Rightarrow y = w}{\Gamma \Rightarrow \langle\lambda x.\phi\rangle(\imath z.\psi)}\ R\lambda_\imath, y!$$

calculi for intuitionistic free logics

- In the context of IL, positive free logics is Scott's IL with existence predicate [4] (aka Beeson's logic of definedness).
- Intuitionistic negative free logics is not (much?) studied.
- $\mathsf{G3ixf}^{\lambda}_{X}$ is the single succedent version of $\mathsf{G3xf}^{\lambda}_{X}$.

Main results

- All rules of $\mathrm{G3i(xf)}_X^\lambda$, but rules $R\vee$, $L\supset, R\exists, L\lambda_\imath^2$, and $R\lambda_\imath$, are hsp-invertible;
- Rules $L\supset$ and $L\lambda_\imath^2$ are hsp-invertible w.r.t. their rightmost premiss only;
- Weakening and contraction are hsp-admissible in $\mathrm{G3i(xf)}_X^\lambda$;
- Cut is syntactically admissible in $\mathrm{G3i(xf)}_X^\lambda$;
- The calculus $\mathrm{G3i(xf)}_X^\lambda$ is sound and complete;
- Maehara's lemma holds in $\mathrm{G3i(xf)}_X^\lambda$, where a partition of the single succedent sequent $\Gamma \Rightarrow, \xi$ n has shape: $\Gamma_1 ; \Gamma_2 \Rightarrow \emptyset ; \xi$.

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