Nested Sequents for Quantified Modal Logics

Tim Lyon (TUDresden) and Eugenio Orlandelli (Unibo)

ExtenDD

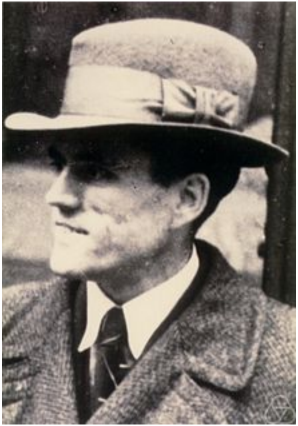
Foreword

- Thanks to Andrzej, Nils, and Yaroslav for the invitation!

- Feel free to interrupt me at any time.

What are sequents? How are they useful?

- **Sequent:** $\Gamma \Rightarrow \Delta$
- Proofs **formalized** as objects in their own right
- Offers **constructive** and **syntactic** approach to studying properties of logics; e.g.
  - Consistency
  - Decidability
  - Interpolation
- **Fruitful** approach to **automated reasoning**; e.g.
  - Complexity optimal decision algorithms with witnesses



Gerhard Gentzen (1945)

A Prominent Desideratum: Analyticity

"A proof is **analytic** if it does not go beyond its **subject matter**."

Bernard Bolzano

Our Interpretation: A proof is **analytic** if it only contains **subformulae** of the **conclusion**.

A Jungle of Sequent Formalisms

$$Rxy, Rxz, x : A \Rightarrow y : B, y : C$$

$$\Gamma \Longrightarrow \Delta / \Gamma_1 \Longrightarrow \Delta_1 / \ldots / \Gamma_n \Longrightarrow \Delta_n$$

$$A \Rightarrow G, [\ \Rightarrow B, [C \Rightarrow D], [E \Rightarrow F]]$$

$$A, B \vdash C, D, E$$

$$A \vdash B \mid C, D \vdash E \mid\ \vdash F$$

$$\Gamma_1 \Longrightarrow \Delta_1 \parallel \ldots \parallel \Gamma_n \Longrightarrow \Delta_n$$

$$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$$

$$A, \overset{1}{[} B, C \overset{1}{]}, \overset{3}{[} D, \overset{2}{[} E \overset{2}{]}, \overset{1}{[} F \overset{13}{]]}$$

Et cetera . . .

The Hierarchy of Sequents

Structure

Gentzen Sequents $A, B \vdash C, D, E$

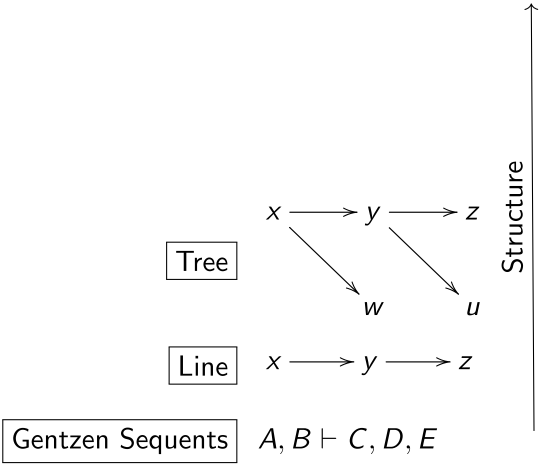
The Hierarchy of Sequents

Structure

Line $x \longrightarrow y \longrightarrow z$

Gentzen Sequents $A, B \vdash C, D, E$

The Hierarchy of Sequents



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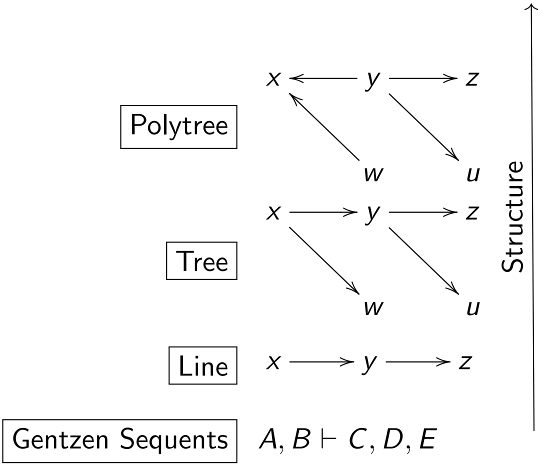

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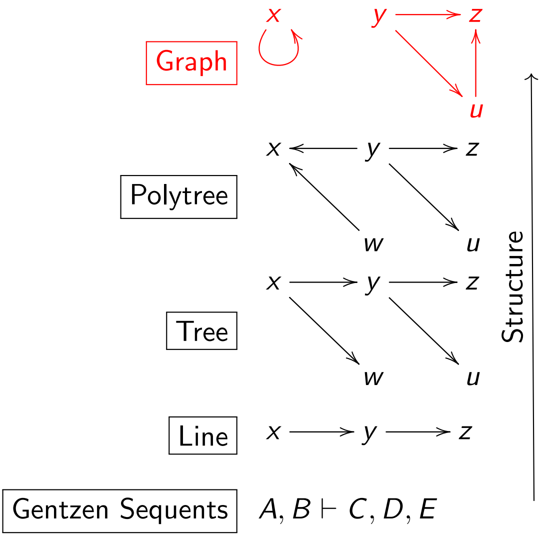
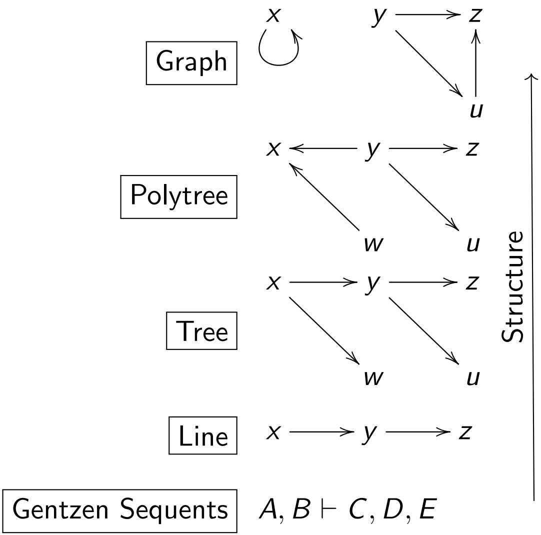


The Hierarchy of Sequents

Q1 Reduce Sequent Structure?



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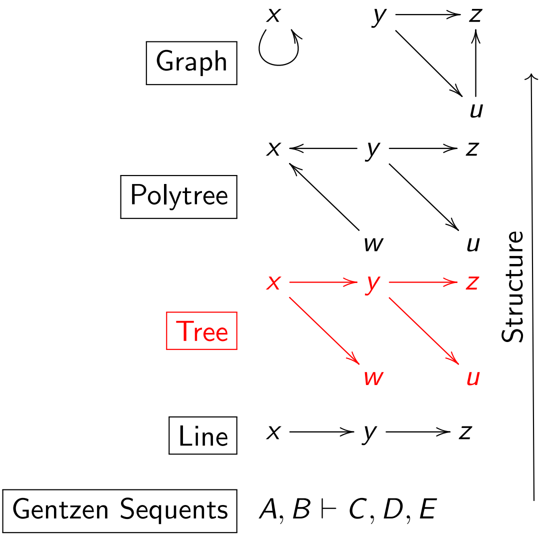
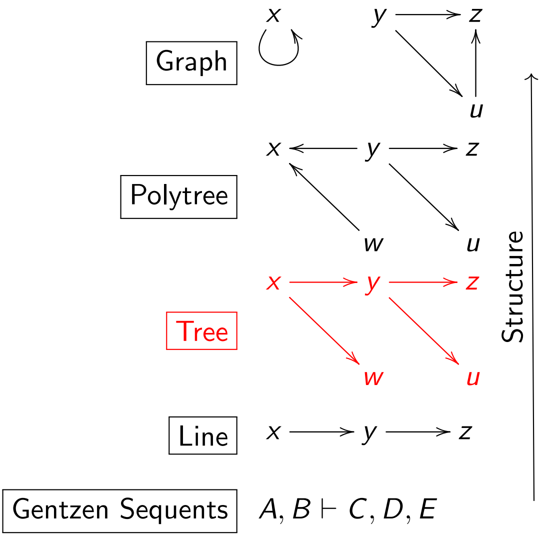


The Hierarchy of Sequents

Q1 Reduce Sequent Structure?

Q2 Retain 'Nice' Properties?

- Invertible Rules
- Admissible Rules
- Syntactic Cut-Elimination



Language:

$$A ::= R(x_1, \ldots, x_n) \mid x = y \mid \bot \mid A \supset A \mid \forall x A \mid \Box A$$
$$\mathcal{E}x \;\equiv\; \exists y(x = y)$$

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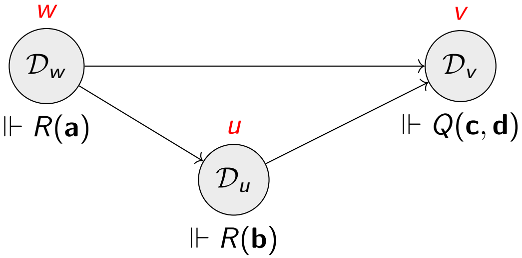
Kripke model with domains: $(\mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{V})$



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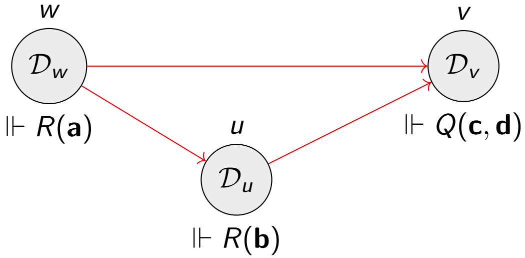

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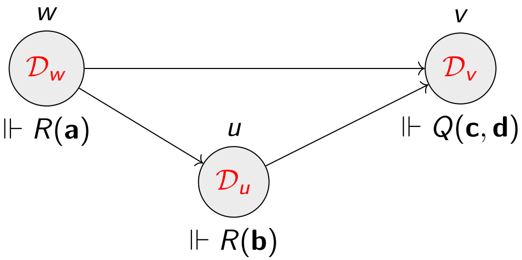

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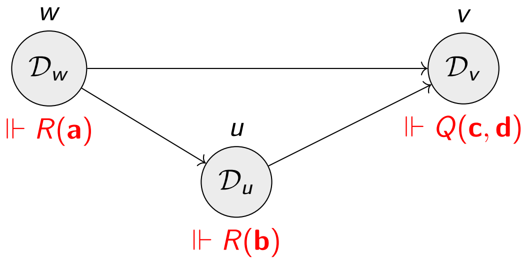

Interpretations

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ *denote objects in* $\mathcal{D}_{\mathcal{W}} = \bigcup_{w \in \mathcal{W}} \{\mathcal{D}_w\}$

$w \Vdash R(\vec{\mathbf{a}})$ means $\vec{\mathbf{a}} \in \mathcal{V}(R, w)$

$w \Vdash \mathbf{a} = \mathbf{b}$ means $\mathbf{a} = \mathbf{b}$

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Base Logic: QK = Set of valid formulae

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Base Logic: QK = Set of valid formulae

Extensions: QK($\mathcal{A}$) = formulae valid on frames satisfying the path and domain conditions in $\mathcal{A}$ (adding seriality).

Path axioms & Relational path conditions

path axiom $\mathsf{G}(n,k)$: $\Diamond^n\Box A \supset \Box^k A$

path condition: $\forall w, v, u(w\mathcal{R}^n v \wedge w\mathcal{R}^k u \supset v\mathcal{R}u)$

Seriality: $\forall w \exists u(w\mathcal{R}u)$

Reflexivity: $\forall w(w\mathcal{R}w)$

Symmetry: $\forall w, u(w\mathcal{R}u \supset u\mathcal{R}w)$

Transitivity: $\forall w, u(w\mathcal{R}u \wedge u\mathcal{R}v \supset u\mathcal{R}v)$

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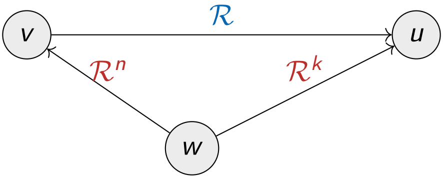

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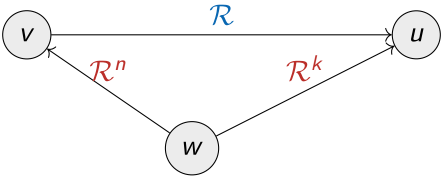

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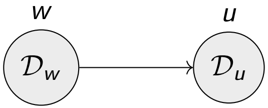
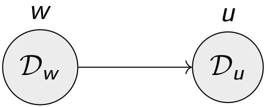
Domain conditions



How are $\mathcal{D}_w$ and $\mathcal{D}_u$ related?

Domain conditions



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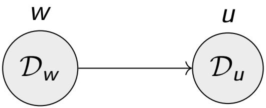

Constant Domains:

$\forall w, u(\mathcal{D}_w = \mathcal{D}_u)$

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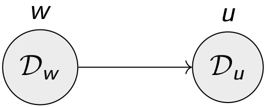

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Increasing Domains:
$\forall w, u(wRu \supset \mathcal{D}_w \subseteq \mathcal{D}_u)$
CBF: $\Box \forall x A \supset \forall x \Box A$

Decreasing Domains:
$\forall w, u(wRu \supset \mathcal{D}_w \supseteq \mathcal{D}_u)$
BF: $\forall x \Box A \supset \Box \forall x A$

Nested Sequents

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$x_1, \ldots, x_\ell; A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k, [\mathcal{S}_1], \ldots, [\mathcal{S}_m]$

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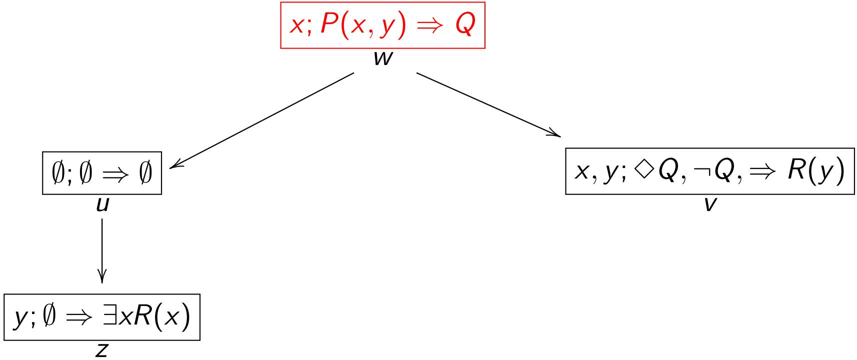

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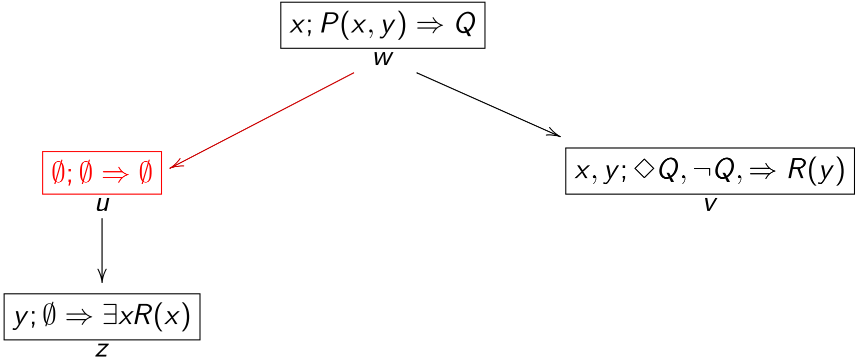

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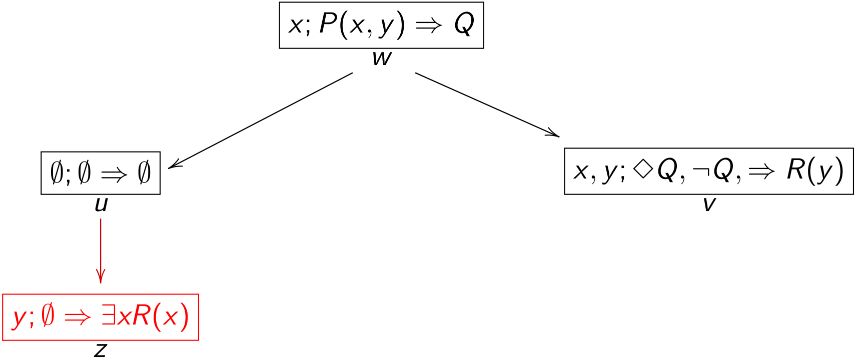

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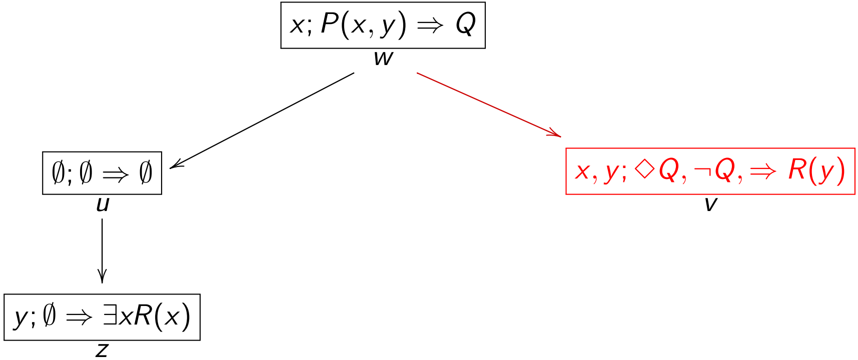

Notation and Interpretation

Notation:

- $X \equiv$ Multiset of Variables $x_1, \ldots, x_\ell$
- $\Gamma, \Delta \equiv$ Multiset of Formulas $A_1, \ldots, A_n$
- $\mathcal{S} \equiv$ Nested Sequent

Interpretation:

$\mathtt{fm}(X; \Gamma \Rightarrow \Delta, [\mathcal{S}_1], \ldots, [\mathcal{S}_m]) = \bigwedge_{x \in X} \mathcal{E}x \wedge \bigwedge_{A \in \Gamma} A \supset \bigvee_{B \in \Delta} B \vee \bigvee_{1 \leq i \leq m} \Box \mathtt{fm}(\mathcal{S}_i)$

Deep rules:

$$\frac{\mathcal{S}\{X'; \Gamma' \Rightarrow \Delta'\}_w}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}_w} R$$

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Nested Calculi: rules for propositional logic and $=$

$$\frac{}{\mathcal{S}\{X; \Gamma, R(\vec{x}) \Rightarrow R(\vec{x}), \Delta\}}\ Ax \qquad \frac{}{\mathcal{S}\{X; \Gamma, \bot \Rightarrow \Delta\}}\ L\bot$$

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow A, \Delta\} \qquad \mathcal{S}\{X; \Gamma, B \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma, A \supset B \Rightarrow \Delta\}}\ L\supset \qquad \frac{\mathcal{S}\{X; \Gamma, A \Rightarrow B, \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow A \supset B, \Delta\}}\ R\supset$$

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$$\frac{\mathcal{S}\{X; P(y/z), x = y, P(x/z), \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; x = y, P(x/z), \Gamma \Rightarrow \Delta\}}\ \mathit{Repl}$$

$$\frac{\mathcal{S}\{X; x = y, \Gamma \Rightarrow \Delta\}\{Y; x = y, \Gamma' \Rightarrow \Delta'\}}{\mathcal{S}\{X; x = y, \Gamma \Rightarrow \Delta\}\{Y; \Gamma' \Rightarrow \Delta'\}}\ \mathit{Rig}$$

Unibo

Quantifier (reachability) rules

$$\frac{\mathcal{S}\{X, y; \Gamma \Rightarrow A(y/x), \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \forall x A, \Delta\}}\ R\forall \text{ (} y \text{ fresh)}$$

$$\frac{\mathcal{S}\{X; \Gamma, \forall x A, A(z/x) \Rightarrow \Delta\}_w\{Y, z; \Gamma' \Rightarrow \Delta'\}_v}{\mathcal{S}\{X; \Gamma, \forall x A \Rightarrow \Delta\}_w\{Y, z; \Gamma' \Rightarrow \Delta'\}_v}\ L\forall,\ (\dagger)$$

Side condition $(\dagger)$: $v = w$ or

- If CBF $\in \mathcal{A}$ then v is backward S4 $\cup\, \mathcal{A}$-reachable from w.
- If BF $\in \mathcal{A}$ then v is forward S4 $\cup\, \mathcal{A}$-reachable from w.
- If UI $\in \mathcal{A}$ then no condition is imposed.

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- If UI $\in \mathcal{A}$ then no condition is imposed.

Modal (propagation) rules

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\emptyset; \emptyset \Rightarrow A]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Box A, \Delta\}} \; R\Box$$

$$\frac{\mathcal{S}\{X; \Gamma, \Box A \Rightarrow \Delta, \}_w\{Y; A, \Gamma' \Rightarrow \Delta'\}_v}{\mathcal{S}\{X; \Gamma, \Box A \Rightarrow \Delta\}_w\{Y; \Gamma' \Rightarrow \Delta'\}_v} \; L\Box \; (\ddagger)$$

Side condition ($\ddagger$): there is an $\mathcal{A}$-path from w to v.

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\emptyset; \emptyset \Rightarrow \emptyset]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}} \; R_D$$

Modal (propagation) rules

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\emptyset; \emptyset \Rightarrow A]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Box A, \Delta\}} \; R\Box$$

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Side condition ($\ddagger$): there is an $\mathcal{A}$-path from w to v.

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\emptyset; \emptyset \Rightarrow \emptyset]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}} \; R_D$$

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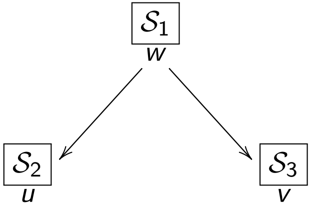
Propagation Rules and Graphs

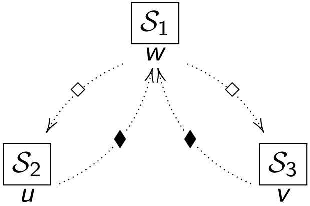
- **Propagation rules**

  1. Nested sequent $\rightsquigarrow$ Propagation graph
  2. Path-axioms & domain conditions $\rightsquigarrow$ Formal grammars
  3. Accepted string $\rightsquigarrow$ Rule application

- **Propagation graphs**

$$\mathcal{S}_1, [\mathcal{S}_2], [\mathcal{S}_3]$$





Propagation Rule Example

If $\mathcal{A}$ is 5 **then** $g(\mathcal{A}) = \{\Diamond \longrightarrow \blacklozenge\Diamond, \blacklozenge \longrightarrow \blacklozenge\Diamond\}$

$$\frac{\Rightarrow \Gamma, [A \Rightarrow], [\Box A \Rightarrow]}{\Rightarrow \Gamma, [\Rightarrow], [\Box A \Rightarrow]} \; L\Box$$

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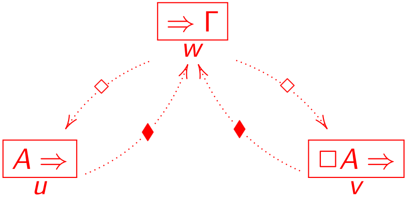

1. Premise $\rightsquigarrow$ propagation graph

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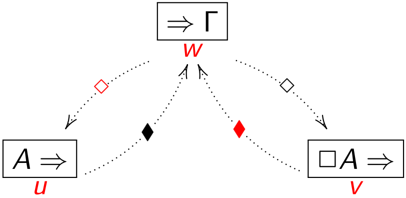

1. Premise $\rightsquigarrow$ propagation graph
2. $\Diamond$ implies a certain path

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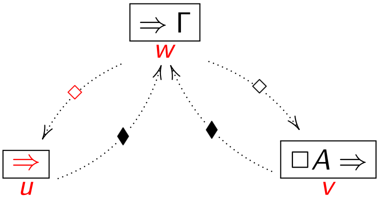

1. Premise $\rightsquigarrow$ propagation graph
2. $\Diamond$ implies a certain path
3. Apply rule

Nice Properties I

1) Height-Preserving Admissibility:

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Pi, \Gamma \Rightarrow \Delta, \Sigma\}} \; IW \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y; \Pi \Rightarrow \Sigma]\}} \; EW$$

$$\frac{\mathcal{S}}{\Rightarrow, [\mathcal{S}]} \; Nec \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y; \Pi_1 \Rightarrow \Delta_1], [Z; \Pi_2 \Rightarrow \Delta_2]\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [Y, Z; \Pi_1, \Pi_2 \Rightarrow \Delta_1, \Delta_2]\}} \; Merge$$

$$\frac{\mathcal{S}\{X; \Gamma, A, A \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma, A \Rightarrow \Delta\}} \; CL \qquad \frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, A, A\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, A\}} \; CR$$

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}} \; SW \qquad \frac{\mathcal{S}\{X, x, x; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X, x; \Gamma \Rightarrow \Delta\}} \; SC$$

Nice Properties II

1) Height-Preserving Admissibility:

$$\frac{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}_w\{X, z; \Gamma' \Rightarrow \Delta'\}_v}{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}_w\{X; \Gamma' \Rightarrow \Delta'\}_v}\ cbf \qquad \frac{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}_w\{X, z; \Gamma' \Rightarrow \Delta'\}_v}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}_w\{X, z; \Gamma' \Rightarrow \Delta'\}_v}\ bf$$

SideC: v is forward S4 $\cup\, \mathcal{A}$-reachable from w & $(C)BF \in \mathcal{A}$.

$$\frac{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}}\ ui$$

$$\frac{\mathcal{S}\{X; \Gamma \Rightarrow \Delta, [\mathcal{S}']\}_w\{Y; \Pi \Rightarrow \Sigma\}_v}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}_w\{(Y; \Pi \Rightarrow \Sigma) \odot \mathcal{S}'\}_v}\ s(\mathcal{A})$$

Side condition: there is a $\mathcal{A}$-path from w to v (same as rule $L\Box$).

Definition (*Fusion operator* $\odot$)

If $\quad \mathcal{S} = Y, \Pi \Rightarrow \Sigma, [\mathcal{S}_1], \dots, [\mathcal{S}_n]$

then $\quad (X; \Gamma \Rightarrow \Delta) \odot \mathcal{S} = X, Y; \Pi, \Gamma \Rightarrow \Delta, \Sigma, [\mathcal{S}_1], \dots, [\mathcal{S}_n]$

Nice Properties II

1) Height-Preserving Admissibility:

$$\frac{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}_w\{X, z; \Gamma' \Rightarrow \Delta'\}_v}{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}_w\{X; \Gamma' \Rightarrow \Delta'\}_v}\ cbf \qquad \frac{\mathcal{S}\{X, z; \Gamma \Rightarrow \Delta\}_w\{X, z; \Gamma' \Rightarrow \Delta'\}_v}{\mathcal{S}\{X; \Gamma \Rightarrow \Delta\}_w\{X, z; \Gamma' \Rightarrow \Delta'\}_v}\ bf$$

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2) Hp-invertible Logical Rules

3) Syntactic Cut-Elimination:

$$\frac{\mathcal{S}\{X;\Gamma \Rightarrow \Delta, A\} \quad \mathcal{S}\{X;A,\Gamma \Rightarrow \Delta\}}{\mathcal{S}\{X;\Gamma \Rightarrow \Delta\}}\ Cut$$

Proof sketch.

- If $A \equiv \forall x B$ is principal we use the hp-admissible rules *cbf*, *bf*, and *ui*.
- If $A \equiv \Box B$ is principal we use the hp-admissible rule $s(\mathcal{A})$

$$\frac{\dfrac{\mathcal{S}\{X;\Gamma \Rightarrow \Delta, [\Rightarrow B]\}\{Y;\Pi \Rightarrow \Sigma\}_v}{\mathcal{S}\{X;\Gamma \Rightarrow \Delta, \Box B\}\{Y;\Pi \Rightarrow \Sigma\}_v}\ R\Box \quad \dfrac{\mathcal{S}\{X;\Box B,\Gamma \Rightarrow \Delta\}_w\{Y;B,\Pi \Rightarrow \Sigma\}_v}{\mathcal{S}\{X;\Box B,\Gamma \Rightarrow \Delta\}_w\{Y;\Pi \Rightarrow \Sigma\}_v}\ L\Box}{\mathcal{S}\{X;\Gamma \Rightarrow \Delta\}_w\{Y,\Pi \Rightarrow \Sigma\}_v}\ Cut$$

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Nice properties IV

4) Soundness & completeness

5) Modularity/analyticity

Scoping non-rigid (and non-denoting) terms

- The sentence

It is necessary that the president of US is a citizen of US.

Is ambiguous between two readings:

  - Joe Biden is necessarily a US-citizen.
  - It is necessary that the US-president, whoever he actually is, is a US-citizen

- Its formal counterpart $\Box Citizen_{US}(\imath x.Pres_{US}(x))$ has only one reading.
- Following [FM98,FM23], we use λ to disambiguate:
  - $\langle\lambda y.\Box Citizen_{US}(y)\rangle(\imath x.Pres_{US}(x))$
  - $\Box(\langle\langle\lambda y.Citizen_{US}(y)\rangle(\imath x.Pres_{US}(x)))$

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Syntax

- Terms:

$$t \ ::= \ x \mid \imath x.A(x)$$

weight of terms: $w(x) = 0, \ w(\imath x.A) = w(A) + 1$

- Formulas:

$$A ::= R(x_1, \ldots, x_n) \mid x = y \mid \bot \mid A \supset A \mid \forall x A \mid \Box A \mid \langle \lambda x.A \rangle (t)$$

weight of λ-formulas: $w(\langle \lambda x.A \rangle (t)) = w(A) + w(t) + 1$

Semantics of $\imath$ and λ

- (Partial) denotation of a $\imath$-term:

$$\mathcal{V}(\imath x.A, w) = \mathbf{o} \quad \textit{iff} \quad \exists! \mathbf{o}' \in \mathcal{D}_{\mathcal{W}}(w \Vdash A(\mathbf{o}') \textit{ and } \mathbf{o}' = \mathbf{o})$$

- Truth for λ:

$$w \Vdash \langle \lambda x.A \rangle (t) \qquad \text{iff} \qquad \exists \mathbf{o} \in \mathcal{D}_{\mathcal{W}}(\mathcal{V}(t, w) = \mathbf{o} \text{ and } w \Vdash A(\mathbf{o}))$$

Lemma (Denotation in the object language)

$w \Vdash \langle \lambda y.y = x \rangle (t)$ iff @w(t and x denote the same object)[a]

[a] Variables (rigidly) denote at each world: $\Vdash \langle \lambda y.y = y \rangle (x)$.

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Semantics of ι and λ

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$$\mathcal{V}(\iota x.A, w) = \mathbf{o} \quad \textit{iff} \quad \exists! \mathbf{o}' \in \mathcal{D}_{\mathcal{W}} (w \Vdash A(\mathbf{o}') \textit{ and } \mathbf{o}' = \mathbf{o})$$

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Unibo

Denotation formulas in nested sequents

- We extend the language with denotation formulas: $t \approx x$ meaning that x (locally) denotes the same object of t.
- A Flat sequent now has shape:

$$X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta$$

where $\overline{\approx_i}$ is a multiset of denotation formulas.

- A nested sequent: $X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta, [\mathcal{S}_1], \ldots, [\mathcal{S}_m]$
- Its formula interpretation:

$$\bigwedge_{x\in X} \mathcal{E}x \wedge \bigwedge_{(t\approx x)\in\overline{\approx_1}} \langle\lambda y.y = x\rangle(t) \wedge \bigwedge_{A\in\Gamma} A \supset \bigvee_{(s\approx z)\in\overline{\approx_2}} \langle\lambda y.y = z\rangle(s) \vee \bigvee_{B\in\Delta} B \vee \bigvee_{1\leq i\leq m} \Box\mathtt{fm}(\mathcal{S}_i)$$

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Rules for denotation formulas: variables

- New initial sequents: $\mathcal{S}\{X; \overline{\approx_1}, y \approx x; \Gamma \Rightarrow \overline{\approx_2}, y \approx x; \Delta\}$
- Rules for variables:

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, x \approx x; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}\ Den_x$$

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, y \approx x; y = x, \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}, y \approx x; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}\ L\approx_x$$

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Rules for denotation formulas: ι-terms

- Denotation for ι-terms:

$$\mathcal{V}(\iota x.A, w) = \mathbf{o} \;\; \textit{iff} \;\; w \Vdash A(\mathbf{o}) \textit{ and } \forall \mathbf{o}' \in \mathcal{D}_{\mathcal{W}}(w \Vdash A(\mathbf{o}') \supset \mathbf{o}' = \mathbf{o})$$

- Rules for ι-terms (obtained from the above clause)

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, \iota x.A \approx y; A(y/x), \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}, \iota x.A \approx y; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}} \; L\approx_1$$

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, \iota x.A \approx y; \Gamma \Rightarrow \overline{\approx_2}; \Delta, A(z/x)\} \quad \mathcal{S}\{X; \overline{\approx_1}, \iota x.A \approx y; z = y, \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}, \iota x.A \approx y; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}} \; L\approx_2$$

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Rules for denotation formulas: $\imath$-terms

- Denotation for $\imath$-terms:

$$\mathcal{V}(\imath x.A, w) = \mathbf{o} \;\; \textit{iff} \;\; w \Vdash A(\mathbf{o}) \; \textit{and} \; \forall \mathbf{o}' \in \mathcal{D}_{\mathcal{W}}(w \Vdash A(\mathbf{o}') \supset \mathbf{o}' = \mathbf{o})$$

- Rules for $\imath$-terms (obtained from the above clause)

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, \imath x.A \approx y; A(y/x), \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}, \imath x.A \approx y; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}} \; L\approx_1$$

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, \imath x.A \approx y; \Gamma \Rightarrow \overline{\approx_2}; \Delta, A(z/x)\} \quad \mathcal{S}\{X; \overline{\approx_1}, \imath x.A \approx y; z = y, \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}, \imath x.A \approx y; \Gamma \Rightarrow \overline{\approx_2}; \Delta\}} \; L\approx_2$$

$$\frac{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta, A(y/x)\} \quad \mathcal{S}\{X; \overline{\approx_1}; \Gamma, A(z/x) \Rightarrow \overline{\approx_2}; \Delta, z = y\}}{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}, \imath x.A \approx y; \Delta\}} \; R\approx, \; z \text{ fresh}$$

Rules for λ-formulas

- Truth for λ-formulas:

$$w \Vdash \langle \lambda x.A \rangle (t) \qquad \text{iff} \qquad \exists \mathbf{o} \in \mathcal{D}_{\mathcal{W}} (\mathcal{V}(t, w) = \mathbf{o} \text{ and } w \Vdash A(\mathbf{o}) \,)$$

- Rules for λ[1]

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, t \approx z; A(z/x), \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}; \langle \lambda x.A \rangle (t), \Gamma \Rightarrow \overline{\approx_2}; \Delta\}} \; L\lambda, z \text{ fresh}$$

$$\frac{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}, t \approx y; \Delta, \langle \lambda x.A \rangle (t)\} \quad \mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta, \langle \lambda x.A \rangle (t), A(y/x)\}}{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta, \langle \lambda x.A \rangle (t)\}} \; R\lambda$$

[1] Similar to those for $\exists$

Rules for λ-formulas

- Truth for λ-formulas:

$$w \Vdash \langle \lambda x.A \rangle (t) \qquad \text{iff} \qquad \exists \mathbf{o} \in \mathcal{D}_{\mathcal{W}}(\mathcal{V}(t,w) = \mathbf{o} \text{ and } w \Vdash A(\mathbf{o})\,)$$

- Rules for λ[1]

$$\frac{\mathcal{S}\{X; \overline{\approx_1}, t \approx z; A(z/x), \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}{\mathcal{S}\{X; \overline{\approx_1}; \langle \lambda x.A \rangle (t), \Gamma \Rightarrow \overline{\approx_2}; \Delta\}}\ L\lambda, z \text{ fresh}$$

$$\frac{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}, t \approx y; \Delta, \langle \lambda x.A \rangle (t)\} \quad \mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta, \langle \lambda x.A \rangle (t), A(y/x)\}}{\mathcal{S}\{X; \overline{\approx_1}; \Gamma \Rightarrow \overline{\approx_2}; \Delta, \langle \lambda x.A \rangle (t)\}}\ R\lambda$$

[1] Similar to those for $\exists$

Main results (wip)

- All (hp-)admissibility results of calculi without $\imath, \lambda$;
- Weakening and contraction of denotation formulas are hp-admissible;
- The rules for $\approx$ and λ are hp-invertible;
- Soundness and completeness.

Future Work

1. Generalize to Cover Wider Classes of Logics
   - Bigger Class of Frame Conditions
2. Prove nice properties (CIP, decidable fragments...)
3. Relationships with Other Calculi, e.g. Labelled and axiomatic

Thanks!