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THE DEDUCTION THEOREMS VALID IN CERTAIN FRAGMENTS OF THE LEWIS' SYSTEM $S2$ AND THE SYSTEM OF FEYS-VON WRIGHT

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1. Introduction

The first deduction theorem valid in modal logic was given by R. C. Barcan in [1] (cf. also [2]). This theorem holds in the Lewis' systems $S4$ and $S5$. A weaker version of this theorem holds in the system $S3$ of Lewis (cf. [5] and [6]). Barcan's theorem does not hold in the Lewis' system $S2$. In [1] Barcan put in fact a question how to modify the classical deduction theorem in order to prove it in the Lewis' system $S2$. In the present paper an answer to this question is given. Namely, we formulate here certain still weaker version of the Barcan's theorem and we prove that it holds in the Lewis' system $S2$. Incidentally we find a modification of the classical deduction theorem which holds in the system T of Feys – von Wright. To put it more precisely, the above mentioned deduction theorems hold in the implicational-conjunctional and in the implicational-conjunctional-negational subsystems of these systems when detachment and adjunction are the only rules of deduction. Notice that neither the set of the implicational-conjunctional, nor the set of the implicational-conjunctional-negational theorems of the systems $S2$ and T respectively, can be axiomatized by means of detachment and adjunction as the rules of deduction.

The question how to modify the classical deduction theorem in order to prove it in Lewis' system $S1$ remains still open.

By the symbols Cxy, Kxy, Nx we shall, respectively, denote implication, conjunction and negation which are formed with the propositional formulas x and y. The set of all thus construed formulas will be denoted by S. The symbol S^c is used to stand for the set of all implications included in S.

Let m, n, k be arbitrary non-negative integers. We shall define an n-argument conjunction

$$K^n_m(x_i)$$

as follows:

(1°) $K^1_m(x_i) = x_m$
(2°) $K^{n+1}_m(x_i) = KK^n_m(x_i)x_{m+n}$

We put, for convenience

(3°) $K^n(x_i) = K^n_1(x_i)$

Further, we shall define an $(n+1)$-argument function (cf. [4])

$$C^n_k(x_i, y)$$

as follows:

(1°) $C^1_k(x_i, y) = Cx_k y$
(2°) $C^{n+1}_k(x_i, y) = CC^n_k(x_i, x_{n+k})C^n_k(x_i, y)$

We put, for convenience

(3°) $C^n(x_i, y) = C^n_1(x_i, y)$

Examples:

$$C^2(x_i, y) = CCx_1x_2Cx_1y$$
$$C^3(x_i, y) = CCCx_1x_2Cx_1x_3CCx_1x_2Cx_1y$$
$$C^4(x_i, y) = CCCCx_1x_2Cx_1x_3CCx_1x_2Cx_1x_4CCCx_1x_2Cx_1x_3CCx_1x_2Cx_1y$$

Let us observe that in accordance with the above definition

$$C^{n-1}(x_i, y) = C^n_2(Cx_1x_i, Cx_1y) =$$
$$= CC^n(x_i, x_{n-1})CC^{n-1}(x_i, x_n) \dots CC^2(x_i, x_3)CC^1(x_i, x_2)C^1(x_i, y)$$

We shall make use of the following rules of deduction:

$r_1 = \{\langle x, Cxy, y\rangle \in S^3\}$
$r_2 = \{\langle x, y, Kxy\rangle \in S^3\}$
$r_3 = \{\langle x, Cyx\rangle \in S^2\}$
$r_4 = \{\langle Cxy, CCzuCxy\rangle \in S^2\}$
In turn, we shall define the consequence function Cn.
Let

$$R \subset \bigcup_{i=2} S^i$$

We put:

$Cn(R : X) = \{x \in S :$ there exist $x_1, x_2, \dots, x_n \in S$ such that $x_n = x$, and for arbitrary $i \leqslant n$, either x_i X, or there exists $r \in R$, and there exist $x_{i_1}, x_{i_2}, \dots, x_{i_k} \in S$ such that

$$\langle x_{i_1}, x_{i_2}, \dots, x_{i_k}\rangle \in R\}$$

The following formulas will be used in the sequel:

$(C1)$ Cxx
$(K1)$ $CKCxCyzCxyCxz$
$(K2)$ $CKCxyCyzCxz$
$(K3)$ $CCxKYzCxy$
$(K4)$ $CCxKyzCxz$
$(K5)$ $CKCxyCxzCxKyz$
$(N1)$ $CxNKyNy$
$(N2)$ $CCkxNyzCKxNzy$
for any $x, y, z \in S$.

2. Direct deduction theorem valid in the system $S2$ of Lewis

THEOREM 1. *Let $(C1)$, $(K1)$, $(K2)$, ..., $(K5) \in X$. Then the following condition is fulfilled:*
(α) *if*

(i) there exists $i \leqslant n$ such that $X + \{x_1\} \subset S^c$,
(ii) $Cn(r_4 : X) \subset X$,
(iii) $y \in Cn(r_1, r_2, : X + \{x_1, x_2, \dots, x_n\})$, then $CK^n(x_i)y \in Cn(r_1, r_2 : X)$.

Condition (α) of Theorem 1 is called the conjunctional theorem on direct deduction. The theorem itself is a modification of the classical deduction theorem of Tarski-Herbrand which states that if

$$y \in Cn(r_1 : X + \{x\}),$$

then

$$Cxy \in Cn(r_1 : X).$$

The deduction theorem (α) holds in the system $S2$ of Lewis. To put it more precisely (α) holds in the implicational-conjunctional sybsystem of the system $S2$ of Lewis based on detachment and adjunction as the only rules of deduction.

3. Direct deduction theorem valid in the system T of Feys-von Wright

THEOREM 2. *Let* $(C1)$, $(K1)$, $(K2)$, …, $(K5) \in X$. *Then the following condition is fulfilled:*
(β) *if*

(i) $Cn(r_3 : X) \subset X$,
(ii) $Cn(r_4 : X) \subset X$,
(iii) $y \in Cn(r_1, r_2, : X + \{x_1, x_2, \ldots, x_n\})$, *then* $CK^n(x_i)y \in Cn(r_1, r_2 : X)$.

The condition (β) is the conjunctional theorem on direct deduction valid in the system T of Feys-von Wright or, more precisely, (β) holds in the implicational-conjunctional subsystem of the system T of Feys-von Wright based on detachment and adjunction as the only rules of deduction.

4. The implicational theorem on direct deduction valid in the system $S2$ of Lewis and in the system T of Feys-von Wright

LEMMA. *Let* $(C1)$, $(K1)$, $(K2)$, …, $(K5) \in X$, *and let one of the two subsequent conditions be satisfied:*

(i) $X \subset S^c$ *and* $Cn(r_4 : X) \subset X$

or
(ii) $Cn(r_3 : X) \subset X$

Then for an arbitrary n, if $CK^n(x_i)y \in Cn(r_1, r_2 : X)$, then $C^n(x_i, y) \in Cn(r_1, r_2 : X)$.

From Theorem 1 and 2, and from the lemma there immediately follows the subsequent:

THEOREM 3. *Let* $(1), (K1), (K2), \ldots, (K5) \in X$. *Then the following conditions are fulfilled:*

(γ) *If*
(i) there exists $i \leqslant n$ *such that* $X + \{x_i\} \subset S^c$,
(ii) $Cn(r_4, X) \subset X$
(iii) $y \in Cn(r_1, r_2 : X + \{x_1, x_2, \ldots, x_n\})$, *then* $C^n(x_i, y) \in Cn(r_1, r_2 : X)$.

(δ) *If*
(i) $Cn(r_3 : X) \subset X$,
(ii) $y \in Cn(r_1, r_2 : X - \{x_1, x_2, \ldots, x_n\})$ *then* $C^n(x_i, y) \in Cn(r_1, r_2 : X)$.

The conditions (γ) and (δ) are the implicational theorems on direct deduction valid, respectively, in the systems $S2$ of Lewis and in the system T of Feys-von Wright.

5. Indirect deduction theorems valid in the system $S2$ of Lewis and in the system T of Feys-von Wright

THEOREM 4. *Let* $(C1), (K1), (K2), \ldots, (K5), (N1), (N2) \in X$. *Then the following conditions are fulfilled:*

(ε) *If*
(i) there exists $i \leqslant n$ *such that* $X + \{y_i\} \subset S^c$,
(ii) $Cn(r_4 : X) \subset X$,
(iii) $x, Nx \in Cn(r_1, r_2 : X + \{y_1, y_2, \ldots, y_n, Nz\})$ *then* $CK^n(y_i)z \in Cn(r_1, r_2 : X)$

(ζ) *If*
(i) $Cn(r_3 : X) \subset X$,
(ii) $x, Nx \in Cn(r_1, r_2 : X + \{y_1, y_2, \ldots, y_n, Nz\})$ *then* $CK^n(y_i)z \in Cn(r_1, r_2 : X)$.

The above lemma and Theorem 4 entail immediately the following:

THEOREM 5. *Let* $(C1), (K1), (K2), \ldots\ldots\ldots, (K5), (N1), (N2) \in X$. *Then the following conditions are fulfilled:*

(η) *If*

(i) *there exists* $i \leqslant n$ *such that* $X + \{y_i\} \subset S^c$,

(ii) $Cn(r_4 : X) \subset X$,

(iii) $x, Nx \in Cn(r_1, r_2 : X + \{y_1, y_2, \ldots, y_n, Nz\})$ *then* $C^n(y_i, z) \in Cn(r_1, r_2 : X)$.

(θ) *If*

(i) $Cn(r_3 : X) \subset X$,

(ii) $x, Nx \in Cn(r_1, r_2 : X + \{y_1, y_2, \ldots, y_n, Nz\})$, *then* $C^n(y_i, z) \in Cn(r_1, r_2 : X)$.

The conditions (ε) and (ζ) are called the conjunctional and the conditions (η) and (θ) are called the implicational theorems on indirect deduction. They amount to modifications of the indirect deduction theorem which was given in [7]: that theorem states that if x, $Nx \in Cn(r_1 : X + \{y_1, y_2, \ldots, y_n, Nz\})$, then $Cy_1Cy_2 \ldots Cy_nz \in Cn(r_1 : X)$. Our theorems ($\varepsilon$) and ($\eta$) hold in the system $S2$ of Lewis while the theorem (ζ) and (θ) hold in the system T of Feys-von Wright.

It is worth while to notice that the condition

$$Cn(r_4 : X) \subset X$$

which appears in the statement of the deduction theorems (α), (γ), (ε) and (η) is essential. Without this condition fulfilled none of these theorems holds in the system $S2$ of Lewis. Likewise, without the condition

$$Cn(r_3 : X) \subset X$$

fulfilled none of the deduction theorems (β), (δ), (ζ) and (θ) holds in the system T of Feys-von Wright.

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