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Janusz Czelakowski

RELATIVELY CONGRUENCE-DISTRIBUTIVE SUBQUASIVARIETIES OF FILTRAL VARIETIES

1 Introduction

Any class of algebras defined by a set of quasi-identities is called a quasivariety. If $\mathbf{K}$ is a class of similar algebras such that $\mathbf{Q}$ is the smallest quasivariety including $\mathbf{K}$, then $\mathbf{K}$ is said to generate $\mathbf{Q}$, and we write $\mathbf{Q} := Qv\mathbf{K}$. ($Va\mathbf{K}$ denotes the variety generated by $\mathbf{K}$.) By well-known results of A. Malcev, G. Grätzer and H. Lakser (see for instance [3]), $Qv\mathbf{K} = ISP_R(\mathbf{K}) = ISPP_U(\mathbf{K})$ for any class $\mathbf{K}$ of algebras, where I, S, P, P_R and P_U denote respectively the operations of forming isomorphic images, subalgebras, direct products, reduced products and ultraproducts.

Let $\mathbf{Q}$ be any quasivariety and A – any algebra, not necessarily in $\mathbf{Q}$. A congruence $\Phi \in ConA$ is called a $\mathbf{Q}$-congruence iff $A/\Phi \in \mathbf{Q}$. $Con_Q A$ denotes the set of all $\mathbf{Q}$-congruences. $Con_Q A$ always contains the universal congruence ∇_A, and it contains the identity congruence Δ_A iff $A \in \mathbf{Q}$. Moreover $Con_Q A$ is an algebraic lattice with respect to $\subseteq$.

A non-trivial algebra A is said to be (*finitely*) *subdirectly irreducible relative to* $\mathbf{Q}$ iff it is not isomorphic to a subdirect product of a (finite) system $B_i, i \in I$, of algebras in $\mathbf{Q}$, unless at least one of them is isomorphic to A; in symbols: $A \cong A' \subseteq_{SD} \Pi_{i\in I} B_i$ with $B_i \in \mathbf{Q}$ (and I finite) only if $A \cong B_i$ for some i. If $\mathbf{Q}$ is clear from the context we say that A is *relatively* (*finitely*) *subdirectly irreducible.*

$A \in \mathbf{Q}$ is relatively (finitely) subdirectly irreducible iff Δ_A (= the diagonal of A) is (finitely) meet-irreducible in $Con_Q A$. Since $Con_Q A$ is algebraic, every $\mathbf{Q}$-congruence on A is the meet of finitely meet-irreducible

$\mathbf{Q}$-congruences. Consequently, every algebra of $\mathbf{Q}$ is isomorphic to a subdirect product of relatively subdirectly irrducible members of $\mathbf{Q}$.

The class of all relatively (finitely) subdirectly irreducible algebras of $\mathbf{Q}$ is denoted by $\mathbf{Q}_{RSI}$ ($\mathbf{Q}_{RFSI}$, respectively).

If $\mathbf{V}$ is a variety of algebras of type τ, every member of $\mathbf{V}_{RSI}$ is *absolutely subdirectly irreducible* (*absolutely finitely subdirectly irreducible*) in the class of all τ-algebras; in this case instead of $\mathbf{V}_{RSI}$ ($\mathbf{V}_{RFSI}$) we simply write $\mathbf{V}_{SI}$ ($\mathbf{V}_{FSI}$, respectively) omitting the subscript R.

LEMMA 1. ([4, Thm. III.8], [6, Lemma 1.5]) *Let* $\mathbf{K}$ *be a class of similar algebras. Then*

$$(Qv\mathbf{K})_{RFSI} \subseteq ISP_U(\mathbf{K}). \ \Box$$

A quasivariety $\mathbf{Q}$ is *relatively congruence-distributive* (RCD, for short) iff $Con_Q A$ is distributive, for all $A \in \mathbf{Q}$.

THEOREM 2. ([7]). *If a quasivariety* $\mathbf{Q}$ *is RCD then*

$$Q_{RFSI} = \mathbf{Q} \cap (Va\mathbf{Q})_{FSI},$$

i.e. every relatively finitely subdirectly irreducible member of $\mathbf{Q}$ *is finitely subdirectly irreducible in the absolute sense.* $\Box$

For a detailed discussion of RCD quasivarieties see [6], [7] and [10].

2 Filtral varieties

Let $A_i, i \in I$, be a non-empty family of similar algebras, let F be a filter over I, and let $\Phi_i \in ConA_i$, for all $i \in I$. The *filtered product* of the congruences $\Phi_i, i \in I$, modulo F is the congruence $\Pi_F \Phi_i$ on the direct product $A := \Pi_{i\in I} A_i$ defined as follows: $f \equiv g(mod\ \Pi_F \Phi_i)$ iff

$$\{i \in I : f(i) \equiv g(i)(mod\ \Phi_i)\} \in F.$$

If F is an ultrafilter, $\Pi_F \Phi_i$ is called the ultrafiltered product (modulo F).

Let us notice that the quotient algebra $\Pi_{i\in I} A_i / \Pi_F \Delta_i$ is identical with the reduced product $\Pi_F A_i$. (Δ_i is the identity congruence on A_i.)

A variety $\mathbf{V}$ of algebras is called *filtral* iff it is semi-simple and for any $A \in V$, any subdirect representation $A \subseteq_{SD} \Pi_{i\in I} A_i$, where $A_i \in \mathbf{V}_{SI}, i \in$

I, and any $\Phi \in ConA$ there is a filter F over I such that $\Phi = A^2 \cap \Pi_F \Delta_i$, where Δ_i is the diagonal of A_i. (The algebra A_i is identified here with its subdirect representations.)

Here are basic facts about filtral varieties.

THEOREM 3. *(i)* ([8, Thm. 5.7]). *Let* $\mathbf{V}$ *be a semi-simple variety. Then* $\mathbf{V}$ *is filtral iff* $\mathbf{V}$ *has equationally definable congruences in the restricted sense.*

(ii) ([9, Cor. 6]). *If* $\mathbf{V}$ *if filtral then* $\mathbf{V}$ *is congruence-distributive and has the congruence extension property (CEP).*

(iii) ([1]). *If* $\mathbf{V}$ *is filtral then* $ISP_U(\mathbf{V}_{SI}) = \mathbf{V}_{SI}$*, i.e.* $\mathbf{V}_{SI}$ *is a universal class.*

(iv) If $\mathbf{V}$ *is filtral then* $\mathbf{V}_{FSI} = \mathbf{V}_{SI}$.

PROOF. We shall only check condition (iv). Since $\mathbf{V} = Qv\mathbf{V}_{SI}$, Lemma 1 and condition (iii) imply that $\mathbf{V}_{FSI} \subseteq ISP_U(\mathbf{V}_{SI}) = \mathbf{V}_{SI}$. Since always $\mathbf{V}_{SI} \subseteq \mathbf{V}_{FSI}$, the equality $\mathbf{V}_{SI} = \mathbf{V}_{FSI}$ follows. □

The following theorem characterizes RCD subquasivarieties of filtral varieties.

THEOREM 4. *Let* $\mathbf{V}$ *be a filtral variety and let* $\mathbf{Q}$ *be a quasivariety contained in* $\mathbf{V}$. *Then* $\mathbf{Q}$ *is* RCD *iff* $\mathbf{Q}$ *is a variety.*

PROOF. The $\Rightarrow$-part is obvious.

($\Leftarrow$). Assume that $\mathbf{Q}$ is RCD. By Theorem 2, $\mathbf{Q}_{RFSI} = \mathbf{Q} \cap (Va\mathbf{Q})_{FSI}$. $Va\mathbf{Q}$, a subvariety of $\mathbf{V}$, is also filtral. This implies that

$$HP(\mathbf{Q} \cap (Va\mathbf{Q})_{SI}) = IP_R(\mathbf{Q} \cap (Va\mathbf{Q})_{SI}).$$

Using condition (iv) of Theorem 3 we have that $\mathbf{Q} \subseteq SP(\mathbf{Q}_{RFSI}) \subseteq SHP(\mathbf{Q}_{RFSI}) = SHP(\mathbf{Q} \cap (Va\mathbf{Q})_{FSI}) = SHP(\mathbf{Q} \cap (Va\mathbf{Q})_{SI}) = SP_R(\mathbf{Q} \cap (Va\mathbf{Q})_{SI}) \subseteq SP_R(\mathbf{Q}) = \mathbf{Q}$. Thus $\mathbf{Q} = SHP(\mathbf{Q}_{RFSI})$.

Since $\mathbf{V}$ has $CEP, SHP(\mathbf{K}) = HSP(\mathbf{K})$, for any class $\mathbf{K} \subseteq \mathbf{V}$. Putting $\mathbf{K} := \mathbf{Q}_{RFSI}$, we obtain: $\mathbf{Q} = SHP(\mathbf{Q}_{RFSI}) = HSP(\mathbf{Q}_{RFSI}) = H(\mathbf{Q})$. So $\mathbf{Q}$ is a variety. □

REMARK 1. Theorem 4. also directly follows from Theorem 2 above and Corollary 3.7 of [2].

COROLLARY 5. *Let* **V** *be a dual discriminator variety. (In particular, let* **V** *be a discriminator variety.) Then every RCD quasivariety contained in* **V** *is a variety.* □

REMARK 2. We would like to make corrections to [5]. Condition (vi) in Proposition 2.3 of [5] should be deleted. (vi) follows from any of the remaining clauses of this proposition. It is unknown if (vi) implies relative strong point-regularity.

The last sentence of [5] should also be altered. The finitely generated varieties of Wajsberg algebras are not Fregean (except for Boolean algebras). It is true, however, that every quasivariety with the relative congruence extension property contained in a finitely generated variety of Wajsberg algebras is actually a variety.

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Section of Logic
Polish Academy of Sciences
Łódź, Poland