Bisequent Calculi for Neutral Free Logic with Definite Descriptions

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Timetable

- Free Logic (FL) and Definite Descriptions (DD).

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- Bisequent Calculus (BSC).

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- Neutral Free Logic (NFL) in the characterisation of Pavlović and Gratzl.

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- BSC for NFL with identity and DD.

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The main feature: singular terms are free from existential assumptions, i.e. they are not assumed to denote an existing object. On the other hand, in all logics under consideration quantifiers are assumed to have an existential import.

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$(E\exists E)$ if $\vdash \alpha \wedge Ex \rightarrow \beta$, then $\vdash \exists x\alpha \rightarrow \beta$, where x is not free in β
$(E\forall I)$ if $\vdash \beta \wedge Ex \rightarrow \alpha$, then $\vdash \beta \rightarrow \forall x\alpha$, where x is not free in β
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(R') $Et \to t = t.$
(LP) $t_1 = t_2 \wedge \alpha[x/t_1] \to \alpha[x/t_2].$

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5. interpretation of identity.

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4. Indrzejczak and Petrukhin 2024 – extension of Pavlović and Gratzl 2023 with identity and definite descriptions (DD).

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Moreover, we add a restriction: $\imath x\alpha$ contains no other DD inside.

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A neutral structure $\mathcal{S}_{nt}$ is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$, where $\mathcal{D} = a_1, \ldots, b_1, \ldots$ is a countable list of free individual variables, and $\mathcal{I}$ is an interpretation function on $\mathcal{L}$

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- $\mathcal{I}(P^n) \subseteq \mathcal{I}(E)^n$ such that if $\langle s, t \rangle \in \mathcal{I}(=)$, then $\langle \ldots, s_i, \ldots \rangle \in \mathcal{I}(P^n)$ iff $\langle \ldots, t_i, \ldots \rangle \in \mathcal{I}(P^n)$, for any n and any $1 \leqslant i \leqslant n$.

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Note that identity is in principle treated as other predicates but it cannot be defined in this way since domains of models contain just parameters, so it should be defined as a condition on models like in Pavlović and Gratzl 2021.

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In case $\mathcal{I}(\imath x\alpha) = a$, for some $a \in \mathcal{I}(E)$, we say that $\imath x\alpha$ is defined and assume that it satisfies the following condition:

$$\mathcal{I}(\imath x\alpha) = \begin{cases} a \text{ iff } \mathcal{V}^i(\alpha[x/a]) = 1 \text{ and } \mathcal{V}^i(\alpha[x/b]) = 0 \text{ for every } a \neq b \in \mathcal{I}(E); \\ \text{otherwise it is undefined.} \end{cases}$$

Semantics – common clauses:

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$\mathcal{V}^i$ is for $\mathcal{V}^w$ [weak] and $\mathcal{V}^s$ [strong] truth-value assignment on the structure $\langle \mathcal{D}, \mathcal{I} \rangle$, defined as follows:

$$\mathcal{V}^i(Et) = \begin{cases} 1 & \text{iff } t \in \mathcal{I}(E), \\ 0 & \text{iff otherwise;} \end{cases} \tag{1}$$

$$\mathcal{V}^i(P^n(t_1, \ldots, t_n)) = \begin{cases} 1 & \text{iff } \langle t_1, \ldots, t_n \rangle \in \mathcal{I}(P^n), \\ u & \text{iff for some } 1 \leqslant i \leqslant n, t_i \notin \mathcal{I}(E), \\ 0 & \text{iff otherwise;} \end{cases} \tag{2}$$

$$\mathcal{V}^i(\neg\alpha) = \begin{cases} 1 & \text{iff } \mathcal{V}^w(\alpha) = 0, \\ u & \text{iff } \mathcal{V}^w(\alpha) = u, \\ 0 & \text{iff } \mathcal{V}^w(\alpha) = 1; \end{cases} \tag{3}$$

$$\mathcal{V}^i(\forall x\alpha) = \begin{cases} 1 & \text{iff for every } t \in \mathcal{I}(E), \mathcal{V}^w(\alpha[x/t]) = 1, \\ 0 & \text{iff for some } t \in \mathcal{I}(E), \mathcal{V}^w(\alpha[x/t]) = 0, \\ u & \text{iff otherwise.} \end{cases} \tag{4}$$

Neutral Free Logic

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Weak valuation $\mathcal{V}^w$:

$$\mathcal{V}^w(\alpha \wedge \beta) = \begin{cases} 1 & \text{iff } \mathcal{V}^w(\alpha) = 1 \text{ and } \mathcal{V}^w(\beta) = 1, \\ u & \text{iff } \mathcal{V}^w(\alpha) = u \text{ or } \mathcal{V}^w(\beta) = u, \\ 0 & \text{iff otherwise;} \end{cases} \tag{5}$$

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Strong valuation $\mathcal{V}^s$:

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- As a consequence, BSC satisfies ordinary subformula property and purity conditions, i.e. in schemata of rules only one (occurrence of a) connective is involved.

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- A bisequent $\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma$ corresponds to a non-standard sequent $\Gamma; \Pi \Rightarrow \Delta; \Sigma$ used by Pavlović and Gratzl.
- We will call the elements of a bisequent:
  - 1-sequent ($\Gamma \Rightarrow \Delta$, corresponds to the consequence relation in 1-logics) and
  - 2-sequent ($\Pi \Rightarrow \Sigma$, corresponds to the consequence relation in 2-logics).

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Bisequent Calculus

- Induced interpretation of bisequents:
  - $\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma$ is falsified by V^i iff all elements of Γ are true, all elements of Δ are either false or undefined, all elements of Π are either true or undefined and all elements of Σ are false.

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- BSC-NFL is 1-logic; it holds: $\vdash \Gamma \Rightarrow \alpha \mid \Rightarrow$ iff $\Gamma \models \alpha$.
- BSC-NFL is 2-logic; it holds: $\vdash \Rightarrow \mid \Pi \Rightarrow \beta$ iff $\Pi \models \beta$.

Axioms and common propositional rules:

BSC for Neutral Free Logic

Axioms and common propositional rules:

A bisequent $\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma$ is axiomatic iff for some atomic formula φ (including identities and Et), either $\varphi \in \Gamma \cap \Sigma$ or $\varphi \in \Gamma \cap \Delta$ or $\varphi \in \Pi \cap \Sigma$. Moreover, bisequents of the form $Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta, P(t_1, \ldots, t_n) \mid P(t_1, \ldots, t_n), \Pi \Rightarrow \Sigma$ are also axiomatic.

$$(\neg{\Rightarrow}|)\quad \frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma} \qquad ({\Rightarrow}\neg\,|)\quad \frac{\Gamma \Rightarrow \Delta \mid \alpha, \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, \neg\alpha \mid \Pi \Rightarrow \Sigma}$$

$$(|\,\neg{\Rightarrow})\quad \frac{\Gamma \Rightarrow \Delta, \alpha \mid \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta \mid \neg\alpha, \Pi \Rightarrow \Sigma} \qquad (|{\Rightarrow}\neg)\quad \frac{\alpha, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \neg\alpha}$$

Propositional rules specific for strong Kleene's connectives:

BSC for Neutral Free Logic

Propositional rules specific for strong Kleene's connectives:

$$(\wedge\Rightarrow|)\ \frac{\alpha, \beta, \Gamma \Rightarrow \Delta \mid S}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta \mid S} \qquad (\Rightarrow\wedge\,|)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid S \qquad \Gamma \Rightarrow \Delta, \beta \mid S}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta \mid S}$$

$$(|\wedge\Rightarrow)\ \frac{S \mid \alpha, \beta, \Gamma \Rightarrow \Delta}{S \mid \alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad (|\Rightarrow\wedge)\ \frac{S \mid \Gamma \Rightarrow \Delta, \alpha \qquad S \mid \Gamma \Rightarrow \Delta, \beta}{S \mid \Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$(\Rightarrow\rightarrow|)\ \frac{\Gamma \Rightarrow \Delta, \beta \mid \alpha, \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, \alpha \rightarrow \beta \mid \Pi \Rightarrow \Sigma}$$

$$(\rightarrow\Rightarrow|)\ \frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha \quad \beta, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}$$

$$(|\Rightarrow\rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \beta}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha \rightarrow \beta}$$

$$(|\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid \Pi \Rightarrow \Sigma \quad \Gamma \Rightarrow \Delta \mid \beta, \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta \mid \alpha \rightarrow \beta, \Pi \Rightarrow \Sigma}$$

Propositional rules specific for weak Kleene's connectives:

BSC for Neutral Free Logic

Propositional rules specific for weak Kleene's connectives:

$$(|\wedge_w\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta\mid\alpha,\beta,\Pi\Rightarrow\Sigma\qquad\Gamma\Rightarrow\Delta,\alpha\mid\alpha,\Pi\Rightarrow\Sigma\qquad\Gamma\Rightarrow\Delta,\beta\mid\beta,\Pi\Rightarrow\Sigma}{\Gamma\Rightarrow\Delta\mid\alpha\wedge\beta,\Pi\Rightarrow\Sigma}$$

$$(|\Rightarrow\wedge_w)\ \frac{\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma,\alpha,\beta\qquad\alpha,\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma,\beta\qquad\beta,\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma,\alpha}{\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma,\alpha\wedge\beta}$$

$(\Rightarrow\rightarrow_w|)$

$$\frac{\Gamma\Rightarrow\Delta,\beta\mid\alpha,\Pi\Rightarrow\Sigma\qquad\Gamma\Rightarrow\Delta,\alpha\mid\alpha,\Pi\Rightarrow\Sigma\qquad\Gamma\Rightarrow\Delta,\beta\mid\beta,\Pi\Rightarrow\Sigma}{\Gamma\Rightarrow\Delta,\alpha\rightarrow\beta\mid\Pi\Rightarrow\Sigma}$$

$(\rightarrow_w\Rightarrow|)$

$$\frac{\alpha,\beta,\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma\qquad\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma,\alpha,\beta\qquad\beta,\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma,\alpha}{\alpha\rightarrow\beta,\Gamma\Rightarrow\Delta\mid\Pi\Rightarrow\Sigma}$$

BSC for Neutral Free Logic

Rules for quantifiers:

$(\Rightarrow\forall\mid)\quad \dfrac{Ea, \Gamma \Rightarrow \Delta, \alpha[x/a] \mid S}{\Gamma \Rightarrow \Delta, \forall x\alpha \mid S}$ $\qquad (\mid\Rightarrow\forall)\quad \dfrac{Ea, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha[x/a]}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \forall x\alpha}$

$(\forall\Rightarrow\mid)\quad \dfrac{Eb, \forall x\alpha, \alpha[x/b], \Gamma \Rightarrow \Delta \mid S}{Eb, \forall x\alpha, \Gamma \Rightarrow \Delta \mid S}$ $\qquad (\mid\forall\Rightarrow)\quad \dfrac{Eb, \Gamma \Rightarrow \Delta \mid \forall x\alpha, \alpha[x/b], \Pi \Rightarrow \Sigma}{Eb, \Gamma \Rightarrow \Delta \mid \forall x\alpha, \Pi \Rightarrow \Sigma}$

where a is fresh and b, b_i are arbitrary parameters.

BSC for Neutral Free Logic

Rules for existence predicate:

$$(E{\Rightarrow}|)\quad \frac{Et, P[t], \Gamma \Rightarrow \Delta \mid S}{P[t], \Gamma \Rightarrow \Delta \mid S} \qquad (|{\Rightarrow}E)\quad \frac{Et, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, P[t]}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, P[t]}$$

$$(|E{\Rightarrow})\quad \frac{Et_1, \ldots, Et_n, P(t_1, \ldots, t_n), \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta \mid P(t_1, \ldots, t_n), \Pi \Rightarrow \Sigma}$$

$$({\Rightarrow}E|)\quad \frac{Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, P(t_1, \ldots, t_n)}{Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta, P(t_1, \ldots, t_n) \mid \Pi \Rightarrow \Sigma}$$

$$(ETr_1)\ \frac{Et, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta \mid Et, \Pi \Rightarrow \Sigma} \qquad (ETr_2)\ \frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, Et}{\Gamma \Rightarrow \Delta, Et \mid \Pi \Rightarrow \Sigma}$$

Both $P[t]$ and $P(t_1, \ldots, t_n)$ denote atoms or identities but not Et, moreover identities of the form $b = d$ are excluded (we have different rules for them). In $P[t]$ there is at least one occurrence of t and there may be other terms; in $P(t_1, \ldots, t_n)$ there are no other terms.

BSC for Neutral Free Logic

Rules for identity:

$(| \Rightarrow =)$

$$\frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, t \approx s \qquad \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, A[x/t] \qquad A[x/s], \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}$$

$$(= \Rightarrow |)\ \frac{t = t, Et, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{Et, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma} \qquad (E\imath \Rightarrow |)\ \frac{a = d, Ed, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{Ed, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}$$

where $A[x/t]$ is an atom, or identity or Et, $t \approx s$ denotes either $t = s$ or $s = t$, a is a fresh parameter and d is an arbitrary description.

BSC for Neutral Free Logic

Rules for DD (where a is a fresh parameter and $\imath x\alpha$ contains no other DD inside):

BSC for Neutral Free Logic

Rules for DD (where a is a fresh parameter and $\imath x\alpha$ contains no other DD inside):

$(\imath \Rightarrow\mid 1)$ $$\frac{\alpha[x/c], c = \imath x\alpha, Ec, \Gamma \Rightarrow \Delta \mid S}{c = \imath x\alpha, Ec, \Gamma \Rightarrow \Delta \mid S}$$

$(\mid \imath \Rightarrow 1)$ $$\frac{Ec, \Gamma \Rightarrow \Delta \mid \alpha[x/c], c = \imath x\alpha, \Pi \Rightarrow \Sigma}{Ec, \Gamma \Rightarrow \Delta \mid c = \imath x\alpha, \Pi \Rightarrow \Sigma}$$

$(\imath \Rightarrow\mid 2)$
$$\frac{c = \imath x\alpha, Eb, Ec, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha[x/b] \quad b = c, c = \imath x\alpha, Eb, Ec, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}{c = \imath x\alpha, Eb, Ec, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}$$

$(\mid \imath \Rightarrow 2)$
$$\frac{Eb, Ec, \Gamma \Rightarrow \Delta, \alpha[x/b] \mid c = \imath x\alpha, \Pi \Rightarrow \Sigma \quad Eb, Ec, \Gamma \Rightarrow \Delta \mid b = c, c = \imath x\alpha, \Pi \Rightarrow \Sigma}{Eb, Ec, \Gamma \Rightarrow \Delta \mid c = \imath x\alpha, \Pi \Rightarrow \Sigma}$$

$(\mid\Rightarrow \imath)$ $$\frac{Ec, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha[x/c] \quad Ea, Ec, \alpha[x/a], \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, a = c}{Ec, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, c = \imath x\alpha,}$$

$(\Rightarrow \imath \mid)$ $$\frac{Ec, \Gamma \Rightarrow \Delta, \alpha[x/c] \mid \Pi \Rightarrow \Sigma \quad Ea, Ec, \Gamma \Rightarrow \Delta, a = c \mid \alpha[x/a], \Pi \Rightarrow \Sigma}{Ec, \Gamma \Rightarrow \Delta, c = \imath x\alpha \mid \Pi \Rightarrow \Sigma}$$

BSC for Neutral Free Logic

Some results:

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Theorem (Substitution)

If $\vdash_n \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda$ *is derivable (where* n *denotes derivability with height bounded by* n*), then the sequent* $\vdash_n \Gamma[b/c] \Rightarrow \Delta[b/c] \mid \Theta[b/c] \Rightarrow \Lambda[b/c]$ *is likewise derivable.*

By induction on the height of the proof. Note that it is restricted to parameters. □

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By induction on the height of the proof. Note that it is restricted to parameters. □

Theorem (Leibniz Law)

For any formula α*, it holds that* $\vdash t_1 = t_2, \alpha[x/t_1] \Rightarrow \alpha[x/t_2] \mid S$.

By induction on the complexity of α. □

BSC for Neutral Free Logic

Theorem (Soundness)

For any bisequent B, if $\vdash B$, then $\models B$.

Theorem (Validity of Lambert axioms)

$(L^{\rightarrow})$ *and* $(L^{\leftarrow})$ *are valid in* $\mathbf{K_3}$ *and* $\mathbf{K_3^w}$.

Lambert Axiom in BSC-NFL:

The proof of $L^{\leftarrow}$ in strong Kleene logic is as follows, where B stands for provable $Ea \Rightarrow \alpha[x/a] \mid \alpha[x/a] \Rightarrow$:

$$\dfrac{B \qquad \dfrac{\dfrac{Eb \Rightarrow \alpha[x/b] \mid \alpha[x/b] \Rightarrow \qquad Ea, Eb \Rightarrow a = b \mid a = b \Rightarrow}{Ea, Eb \Rightarrow a = b \mid \alpha[x/b], \alpha[x/b] \to a = b \Rightarrow}\,(\mid\to\Rightarrow)}{Ea, Eb \Rightarrow a = b \mid \alpha[x/b], \forall x(\alpha \to a = x) \Rightarrow}\,(\mid\forall\Rightarrow)}{Ea \Rightarrow a = \imath x\alpha \mid \alpha[x/a], \forall x(\alpha \to a = x) \Rightarrow}\,(\Rightarrow\imath\mid)$$

$$\dfrac{\dfrac{\dfrac{Ea \Rightarrow a = \imath x\alpha \mid \alpha[x/a], \forall x(\alpha \to a = x) \Rightarrow}{Ea \Rightarrow a = \imath x\alpha \mid \alpha[x/a] \wedge \forall x(\alpha \to a = x) \Rightarrow}\,(\mid\wedge\Rightarrow)}{Ea \Rightarrow \alpha[x/a] \wedge \forall x(\alpha \to a = x) \to a = \imath x\alpha \mid\Rightarrow}\,(\Rightarrow\to\mid)}{\Rightarrow \forall y(\alpha[x/y] \wedge \forall x(\alpha \to y = x) \to y = \imath x\alpha) \mid\Rightarrow}\,(\Rightarrow\forall\mid)$$

Lambert Axiom in BSC-NFL:

Let D stands for the following proof:

$$\dfrac{Ea \Rightarrow \alpha[x/a] \mid \alpha[x/a] \Rightarrow}{Ea \Rightarrow \alpha[x/a] \mid a = \imath x\alpha \Rightarrow}\ (\mid \imath \Rightarrow 1)$$

Then:

$$\dfrac{D \qquad \dfrac{\dfrac{\dfrac{Eb \Rightarrow \alpha[x/b] \mid \alpha[x/b] \Rightarrow \qquad Ea, Eb \Rightarrow a = b \mid a = b \Rightarrow}{Ea, Eb \Rightarrow a = b \mid \alpha[x/b], a = \imath x\alpha \Rightarrow}\ (\mid \imath \Rightarrow 2)}{Ea, Eb \Rightarrow \alpha[x/b] \rightarrow a = b \mid a = \imath x\alpha \Rightarrow}\ (\Rightarrow \rightarrow \mid)}{Ea \Rightarrow \forall x(\alpha \rightarrow a = x) \mid a = \imath x\alpha \Rightarrow}\ (\Rightarrow \forall \mid)}{\dfrac{\dfrac{Ea \Rightarrow \alpha[x/a] \wedge \forall x(\alpha \rightarrow a = x) \mid a = \imath x\alpha \Rightarrow}{Ea \Rightarrow a = \imath x\alpha \rightarrow \alpha[x/a] \wedge \forall x(\alpha \rightarrow a = x) \mid \Rightarrow}\ (\Rightarrow \rightarrow \mid)}{\Rightarrow \forall y(y = \imath x\alpha \rightarrow \alpha[x/y] \wedge \forall x(\alpha \rightarrow y = x)) \mid \Rightarrow}\ (\Rightarrow \forall \mid)}\ (\Rightarrow \wedge \mid)$$

Lambert Axiom in BSC-NFL

All rules for DD are derivable if we use $(L^{\rightarrow})$ or $(L^{\leftarrow})$ as additional axioms and use cuts which will be proved admissible in the next section.

BSC for Neutral Free Logic

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Theorem (Completeness)

For any bisequent B, if $\models B$, then $\vdash B$.

Cut Admissibility

Preliminary results:

Cut Admissibility

Preliminary results:

Theorem (Generalisation of axioms)

For any formula α, the following bisequents are derivable:

1. $\alpha, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha$,
2. $\alpha, \Gamma \Rightarrow \Delta, \alpha \mid \Pi \Rightarrow \Sigma$,
3. $\Gamma \Rightarrow \Delta \mid \alpha, \Pi \Rightarrow \Sigma, \alpha$,
4. $Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta, \alpha(t_1, \ldots, t_n) \mid \alpha(t_1, \ldots, t_n), \Pi \Rightarrow \Sigma$.

By induction on the complexity of α. □

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3. $\Gamma \Rightarrow \Delta \mid \alpha, \Pi \Rightarrow \Sigma, \alpha$,
4. $Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta, \alpha(t_1, \ldots, t_n) \mid \alpha(t_1, \ldots, t_n), \Pi \Rightarrow \Sigma$.

By induction on the complexity of α. □

Theorem (Invertibility)

All the rules of the bisequent calculi in question are height-preserving invertible.

By induction on the height of the derivation, using Substitution Theorem. □

Cut Admissibility

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Preliminary results:

Theorem (Admissibility of structural rules)

Structural rules of weakening and contraction are height-preserving admissible.

$$(W{\Rightarrow}|)\ \frac{\Gamma \Rightarrow \Delta \mid S}{\alpha, \Gamma \Rightarrow \Delta \mid S} \qquad ({\Rightarrow}W\ |)\ \frac{\Gamma \Rightarrow \Delta \mid S}{\Gamma \Rightarrow \Delta, \alpha \mid S}$$

$$(|\ W{\Rightarrow})\ \frac{S \mid \Pi \Rightarrow \Sigma}{S \mid \alpha, \Pi \Rightarrow \Sigma} \qquad (|{\Rightarrow}W)\ \frac{S \mid \Pi \Rightarrow \Sigma}{S \mid \Pi \Rightarrow \Sigma, \alpha}$$

$$(C{\Rightarrow}|)\ \frac{\alpha, \alpha, \Gamma \Rightarrow \Delta \mid S}{\alpha, \Gamma \Rightarrow \Delta \mid S} \qquad ({\Rightarrow}C\ |)\ \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha \mid S}{\Gamma \Rightarrow \Delta, \alpha \mid S}$$

$$(|\ C{\Rightarrow})\ \frac{S \mid \alpha, \alpha, \Pi \Rightarrow \Sigma}{S \mid \alpha, \Pi \Rightarrow \Sigma} \qquad (|{\Rightarrow}C)\ \frac{S \mid \Pi \Rightarrow \Sigma, \alpha, \alpha}{S \mid \Pi \Rightarrow \Sigma, \alpha}$$

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Theorem (Transfer)

The following rules are height-preserving admissible:

$$(LTr)\quad \frac{\Gamma \Rightarrow \Delta \mid \alpha, \Pi \Rightarrow \Sigma}{\alpha, \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma} \qquad (RTr)\quad \frac{\Gamma \Rightarrow \Delta, \alpha \mid \Pi \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \alpha}$$

Forms of Cut:

Cut Admissibility

Forms of Cut:

$$\text{(E-Cut)}\ \frac{\Gamma \Rightarrow \Delta \mid \Lambda \Rightarrow \Theta, Et \quad Et, \Pi \Rightarrow \Sigma \mid \Xi \Rightarrow \Omega}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid \Lambda, \Xi \Rightarrow \Theta, \Omega}$$

(L-Cut)

$$\frac{Et_1, \ldots, Et_n, \Gamma \Rightarrow \Delta \mid \Lambda \Rightarrow \Theta, P(t_1, \ldots, t_n) \quad Et_1, \ldots, Et_n, P(t_1, \ldots, t_n), \Pi \Rightarrow \Sigma \mid \Xi \Rightarrow \Omega}{Et_1, \ldots, Et_n \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid \Lambda, \Xi \Rightarrow \Theta, \Omega}$$

$$\text{(O-Cut)}\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid \Lambda \Rightarrow \Theta \quad \alpha, \Pi \Rightarrow \Sigma \mid \Xi \Rightarrow \Omega}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid \Lambda, \Xi \Rightarrow \Theta, \Omega}$$

$$\text{(I-Cut)}\ \frac{\Gamma \Rightarrow \Delta \mid \Lambda \Rightarrow \Theta, \alpha \quad \Pi \Rightarrow \Sigma \mid \alpha, \Xi \Rightarrow \Omega}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid \Lambda, \Xi \Rightarrow \Theta, \Omega}$$

$$\text{(R-Cut)}\ \frac{\Gamma \Rightarrow \Delta \mid \alpha, \Lambda \Rightarrow \Theta \quad \Pi \Rightarrow \Sigma, \alpha \mid \Xi \Rightarrow \Omega}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid \Lambda, \Xi \Rightarrow \Theta, \Omega}$$

$$\text{(3-Cut)}\ \frac{\alpha, \Gamma_1 \Rightarrow \Delta_1 \mid \Lambda_1 \Rightarrow \Theta_1 \quad \Gamma_2 \Rightarrow \Delta_2 \mid \Lambda_2 \Rightarrow \Theta_2, \alpha \quad \Gamma_3 \Rightarrow \Delta_3, \alpha \mid \alpha, \Lambda_3 \Rightarrow \Theta_3}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3 \mid \Lambda_1, \Lambda_2, \Lambda_3 \Rightarrow \Theta_1, \Theta_2, \Theta_3}$$

Cut Admissibility

Hypothesis: Two cuts are enough.

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Cut Admissibility

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Theorem (Cut admissibility)

The rules (E-Cut), (L-Cut), (O-Cut), (I-Cut), (R-Cut), (3-Cut) are admissible.

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Proved simultaneously by double induction on the complexity of the cut formula and on the height of the cut (the sum of heights of premises of cut).
Proof in Pavlović and Gratzl 2023; extended to identity and DD by Indrzejczak and Petrukhin 2024.

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  2. $\Rightarrow \alpha \mid \Gamma \Rightarrow$ corresponds to the liberal consequence which leads from non-falsity to truth.
- The application of BSC may be extended also to paraconsistent versions of neutral free logics and to theories of DD built on **FDE**.

Thank you for attention!

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